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MULTISCALE APPROXIMATION OF SURFACE AND INTERNAL WAVES INTERACTION IN TWO-LAYER FLUID

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Interaction of internal and surface waves in two-layer fluid with free surface is considered. The amplitudes of the second harmonics of elevations of the contact surface and the free surface are investigated for two pairs of frequencies of the center of the wave packet. It is shown, that the wave crest moves faster than the base and collapses, while then it becomes smoother due to dispersion. The effect is determined by the influence of nonlinearity and dispersion on propagation of the internal and surface waves.

KEY WORDS: surface waves, internal waves, two-layer fluid.

МНОГОМАСШТАБНАЯ АППРОКСИМАЦИЯ ВЗАИМОДЕЙСТВИЯ ПОВЕРХНОСТНЫХ И ВНУТРЕННИХ ВОЛН В ДВУХСЛОЙНОЙ ЖИДКОСТИ

Авраменко О.В., Нарадовый В.В.

Рассматривается взаимодействие внутренних и поверхностных волн в двухслойной жидкости со свободной поверхностью. Исследованы амплитуды второй гармоники элевации контактной поверхности и свободной поверхности для двух пар частот центра волнового пакета. Было показано, что гребень волны движется быстрее, чем основание и сначала схлопывается, а затем сглаживается за счет дисперсии, что связано с влиянием нелинейности и дисперсии на распространение внутренних и поверхностных волн.

КЛЮЧЕВЫЕ СЛОВА: поверхностные волны, внутренние волны, двухслойная жидкость.

БАГАТОМАСШТАБНА АПРОКСИМАЦІЯ ВЗАЄМОДІЇ ПОВЕРХНЕВИХ ТА ВНУТРІШНІХ ХВИЛЬ У ДВОШАРОВИХ РІДИНАХ

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Розглядається взаємодія внутрішніх і поверхневих хвиль у двошаровій рідині з вільною поверхнею. Досліджено амплітуди другої гармоніки елевації контактної поверхні і вільної поверхні для двох пар частот центру хвильового пакета. Було показано, що гребінь хвилі рухається швидше, ніж основа і спочатку схлопується, а потім згладжується за рахунок дисперсії, що пов'язано з впливом нелінійності і дисперсії на поширення внутрішніх і поверхневих хвиль.

КЛЮЧОВІ СЛОВА: поверхневі хвилі, внутрішні хвилі, двошарова рідина.

1. Introduction. The study of wave processes in different hydrodynamic systems is one of the most actual problems of hydrodynamics. Recently more experimentalists have joined to the large number of theoretical works.

Profiles of two dimensional standing gravity waves are investigated experimentally in [5] and [6]. It is shown that the waves of secondary circulation flows covering the entire thickness of fluid take place.

Also, experimental investigations for three-layer fluid system with a thin layer inside and on condition of leak on the layer were carried out and the generation of soliton waves in this system was investigated. It has been shown that there are such parameters when the individual waves are generated at the surface of contact and then spread to the surrounding flow. I.T. Selezov [7] carried out a preliminary analysis of wave propagation in such a system which led to a nonlinear Schrödinger equation similarly to A. Nayfeh [8].

A new stable configuration of the three-phase system formed by gas and fluid layer with a free surface and a layer of fluid with solid bottoms is investigated experimentally and theoretically in [9]. It is shown that in the presence of large surface tension on the free surface with shallow upper layer the local discontinuities of upper layer occur in the surface of the contact of the lower layer with gas.

S.V. Korsunskyy investigated linear and nonlinear surface and internal waves [10] in the framework of the two-layer model stratified environment in full statement. Corresponding evolution equations of KdV and their analysis and comparison with the results of the model of "rigid lid" were received. Boussinesq-type equations for interacting modes of similar and different

types were built.
Regularities of gravitational nonlinear wave motion in two-layer stratified fluid of finite depth for upper lighter layer were investigated in [11]. Features of nonlinear internal resonance interaction of gravitational waves generated by the free surface and the contact surface of fluid environments were reviewed. It was shown that degenerate and secondary combinational resonance interactions were implemented at the calculation of second-order of smallness.

Camassa with co-authors [12] investigated completely nonlinear waves in two-layer fluid with free surface. Completely nonlinear long waves in two-layer fluid were considered in [13].

This article is devoted to study of interaction of the internal and surface waves in the two-layer fluid of finite depth with a free surface.

2. Statement of the problem and method of solution.
The problem of propagation of internal and surface waves of finite amplitude on the surface of the fluid layer $\Omega_1 = \{(x,z) : |x| < \infty, -h_1 \le z < 0\}$ with density ρ_1 and upper fluid layer $\Omega_2 = \{(x,z) : |x| < \infty, 0 \le z \le h_2\}$ with density ρ_2 is considered. The layers are separated by the surface $z = \eta(x,t)$ and the upper layer is bounded by free surface $z = \eta_0(x,t)$. The surface tension on the contact surface and on the free surface has been taken into account. The force of gravity is directed perpendicularly to the interface in the negative z-direction, and both fluids are considered as incompressible ones (Fig. 1).

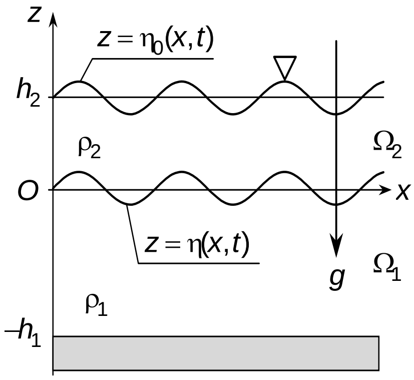


Fig. 1. Problem formulation. A sketch.

Let us introduce dimensionless values by using characteristic length L, the maximal elevation of the free surface a, characteristic time $(L/g)^{1/2}$, density of the lower layer ρ_1, where g is gravitational acceleration. Dimensionless coefficients of surface tension on the free surface T_0 and on the surface of contact T are $\left(T^*, T_0^*\right) = \left(T, T_0\right) / \left(L^2 \rho g\right)$.

Mathematical statement of the problem has the form (asterisks are omitted)

$$\nabla^2 \phi_j = 0 \quad \text{at} \; \Omega_j,$$

$$\eta_{,t} - \phi_{j,z} = -\alpha \phi_{j,x} \eta_{,x} \qquad \text{at} \quad z = \alpha \eta(x,t),$$

$$\eta_{0,t} - \phi_{2,z} = -\alpha \phi_{2,x} \eta_{0,x} \qquad \text{at} \quad z = \alpha \eta_0(x,t),$$

$$\phi_{1,t} - \rho \phi_{2,t} + (1-\rho)\eta + 0.5\alpha\left[\left(\nabla \phi_1\right)^2 - \rho\left(\nabla \phi_2\right)^2\right] -$$

$$-T\left(1 + \alpha^2 \eta_{,x}^2\right)^{-3/2} \eta_{,xx} = 0 \quad \text{at} \; z = \alpha \eta(x,t), \qquad (1)$$

$$\phi_{2,t} + \eta_0 + 0.5\alpha\left(\nabla \phi_2\right)^2 - T_0\left(1 + \alpha^2 \eta_{0,x}^2\right)^{-3/2} \eta_{0,xx} = 0 \; \text{at}$$

$$z = \alpha \eta_0(x,t),$$

$$\phi_{1,z} = 0 \; \text{at} \; z = -h_1,$$

where $j = 1, 2$, $\rho = \rho_2 / \rho_1$ is ratio of densities of the upper and lower layers, $\alpha = a / L$ is the coefficient of nonlinearity.

We assume $\alpha << 1$, so the model describes the weakly nonlinear two-layer system with dispersion. To determine the approximate solution of the problem (1) for small but finite amplitudes, the method of multiple scale expansions up to the third order is used I the form

$$\eta(x,t) = \sum_{n=1}^{3} \alpha^{n-1} \eta_n(x_0, x_1, x_2, t_0, t_1, t_2) + O(\alpha^3)$$

$$\eta_0(x,t) = \sum_{n=1}^{3} \alpha^{n-1} \eta_{0n}(x_0, x_1, x_2, t_0, t_1, t_2) + O(\alpha^3), \qquad (2)$$

$$\phi_j(x,z,t) = \sum_{n=1}^{3} \alpha^{n-1} \phi_{jn}(x_0, x_1, x_2, z, t_0, t_1, t_2) + O(\alpha^3),$$

where $j = 1, 2$, $x_n = \alpha^n x$, $t_n = \alpha^n t$ are scale variables.

Substitution of expansions (2) into the equation (1) leads to three linear problems for the unknown functions η_1, η_{01}, ϕ_{11}, ϕ_{21}, η_2, η_{02}, ϕ_{12}, ϕ_{22}, η_3, η_{03}, ϕ_{31}, ϕ_{32}

3. Analysis of the second approximation problem solutions.
Solutions of the first linear problem and analysis of interaction of the internal and surface waves in the first approximation are given in [1, 2]. In [3] the second linear problem solutions were found and evolution equations for the wave packets envelopes were obtained, as well as the form of the wave packet on the surface of contact and on the free surface were analyzed. In [4] a condition of the modulation stability of wave packets was derived and investigated, the corresponding stability diagrams were constructed. This article presents the analyses of the interaction of wave packets on the contact surface and on the free surface taking into account the second approximation and solutions of evolution equations.

Here some of the results obtained in the previous works are reproduced. The solution of the first linear approximation problem and the dispersion equation are as follows

$$\eta_1 = A e^{i\theta} + \bar{A} e^{-i\theta}, \qquad (3)$$

$$\eta_{01} = \frac{\omega^2 (A e^{i\theta} + \bar{A} e^{-i\theta})}{\omega^2 \operatorname{ch}(kh_2) - (k + T_0 k^3) \operatorname{sh}(kh_2)},$$

$$\omega^2 \operatorname{cth}(kh_1) + \rho\omega^2\left(\frac{\omega^2-(k+T_0k^3)\operatorname{cth}(kh_2)}{\omega^2\operatorname{cth}(kh_2)-(k+T_0k^3)}\right) =$$
$$= (1-\rho)k + Tk^3,$$

where $A(x_1, x_2, t_1, t_2)$ and $\bar{A}(x_1, x_2, t_1, t_2)$ are envelope of the wave packet on the contact surface and the complex conjugate to the envelope of the wave packet *A* and $\theta = kx_0 - \omega t_0$, *k* is wave number, and ω is the frequency of the centre of the wave packet. It worth to note, the dispersion equation has two pairs of solutions unlike previously studied problems of the type. The solutions to the problem of the second linear approximation are

$$\eta_2 = \frac{0.5\omega^2}{1-\rho}\Bigg(1-\rho-\operatorname{cth}^2(kh_1)+$$
$$+\rho\left[\frac{(1-\rho)k+Tk^3-\omega^2\operatorname{cth}(kh_1)}{\rho\omega^2}\right]^2\Bigg)A\bar{A}+ \qquad (4)$$
$$+\Lambda e^{2i\theta}A^2 + cc,$$

$$\eta_{02} = \Lambda_0 e^{2i\theta}A^2 + \frac{0.5\omega^2\left(\omega^4-(k+T_0k^3)^2\right)}{\omega^2\operatorname{ch}(kh_2)-(k+T_0k^3)\operatorname{sh}(kh_2)} + cc,$$

where cc is the complex conjugate value of the preceding expression.

Evolution equations of envelopes on the surface of contact and on the free surface

$$A_{,t} + \omega' A_{,x} - 0.5\omega'' A_{,xx} = i\alpha^2 I A^2 \bar{A},$$
$$A^0_{,t} + \omega' A^0_{,x} - 0.5\omega'' A^0_{,xx} = i\alpha^2 I_0 (A^0)^2 \bar{A}^0,$$

and solutions of evolution equations are of the form

$$A = a\exp(i\alpha^2 a^2 \omega^{-1} It), \qquad (5)$$
$$A^0 = a^0 \exp(i\alpha^2 (a^0)^2 \omega^{-1} I_0 t).$$

In the first approximation the interaction of the internal and surface waves was investigated based on the expression [1]

$$\eta_1 = A_1\cos(kx-\omega_1 t) + A_2^0 a_2 \cos(kx-\omega_2 t),$$
$$\eta_{01} = A_1 a_1 \cos(kx-\omega_1 t) + A_2^0\cos(kx-\omega_2 t), \qquad (6)$$

where

$$a_1 = \frac{\omega_1^{\,2}}{\omega_1^{\,2}\operatorname{ch}(kh_2)-(k+T_0k^3)\operatorname{sh}(kh_2)},$$
$$a_2 = \left(\frac{\omega_2^{\,2}}{\omega_2^{\,2}\operatorname{ch}(kh_2)-(k+T_0k^3)\operatorname{sh}(kh_2)}\right)^{-1}.$$

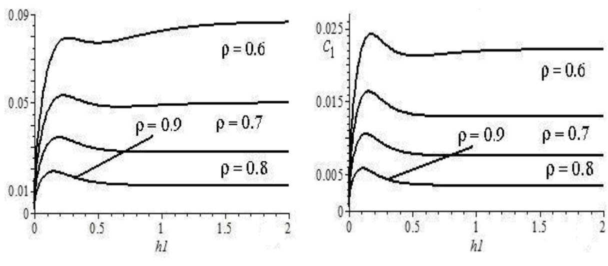
Let us analyze the amplitudes of the second approximation of elevation of the free surface $\eta_0(x,t)$ and of the second approximation of elevation of the contact surface $\eta(x,t)$ corresponding to the pairs of frequencies $\pm 2\omega_1$ and $\pm 2\omega_2$. We denote these ratios c_1 and c_2 respectively

$$c_1 = \frac{\Lambda_0(2\omega_1)}{\Lambda(2\omega_1)}, \quad c_2 = \left(\frac{\Lambda_0(2\omega_2)}{\Lambda(2\omega_2)}\right)^{-1}. \qquad (7)$$

The value c_1 characterizes contribution of the waves with frequency $2\omega_1$ into the surface wave movement and value c_2 characterizes contribution of the waves with frequency $2\omega_2$ into the wave movement at the interface between the two fluid layers.

In Fig. 2 the dependence of variable c_1 on the thickness h_1 of the lower layer at different ratios ρ is presented for the following system parameters: $h_2 = 1$, $\rho \in \{0.6, 0.7, 0.8, 0.9\}$, $T = T_0 = 0$, and the wave numbers $k = 1.5$ (Fig. 2a) and $k = 2$ (Fig. 2b). Variations in ρ affect the value of c_1. Moreover, with increasing the wave number k the value c_1 becomes smaller, although still greater than zero. In each case, there is certain asymptotic value to which c_1 tends with increasing the thickness of the lower layer.

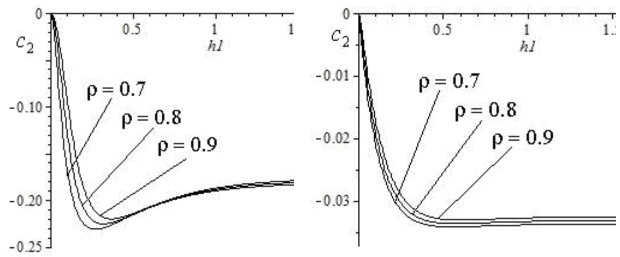
With increasing h_1 from 0 to 0.5 it is noticeable that for each wave number k and ratio ρ the value c_1 reaches a local maximum on this interval. This means that for such a system parameters the contribution of the waves with frequency $2\omega_1$ in the surface movement is the highest possible one for the gravitational waves. In Fig. 3 the dependence of c_2 on the thickness of the lower layer h_1 at different ρ is represented for the following system parameters: $h_2 = 1$, $T = T_0 = 0$, $\rho \in \{0.7, 0.8, 0.9\}$ and the wave numbers $k = 1.5$ (Fig. 3a) and $k = 2$ (Fig. 3b).



a) $k = 1.5$ b) $k = 2$

Fig. 2. Dependences of c_1 *on thickness of the lower layer.*

Changes in the ratio of densities slightly affect the changes in the value c_2, whereby it remains negative at different wave numbers. It should be noted, that analysis of the amplitude ratio in the first approximation reveals the opposite phenomenon, namely the value a_1 corresponding to the wave with frequency ω_1 was always negative and the value a_2 corresponding to wave with frequency ω_2 was positive, and also it almost does not depend on changes in ρ. When h_1 grows from 0 to 0.5, the local minimum at different ρ takes place that was observed for c_1. Therefore, for such system parameters the contribution of the waves with frequency $2\omega_2$ of the wave motion on the contact surface between the two layers is maximal for the capillary waves.



a) $k = 1.5$ b) $k = 2$

Fig. 3. Dependences of c_2 on thickness of the lower layer.

Elevation of the contact surface and elevation of the free surface in the second approximation have the form

$$\eta_2 = 2A^2\left(B(2\omega_1)+\Lambda(2\omega_1)\cos(2kx-2\omega t)\right)+ \\ +2\left(A_2^0\right)^2\left(B(2\omega_2)+c_2\Lambda(2\omega_2)\cos(2kx-2\omega t)\right), \qquad (8)$$

$$\eta_{02} = 2A_1^2\left(C(2\omega_1)+c_1\Lambda_0(2\omega_1)\cos(2kx-2\omega t)\right)+ \\ +2\left(A_2^0\right)^2\left(C(2\omega_2)+\Lambda_0(2\omega_2)\cos(2kx-2\omega t)\right)$$

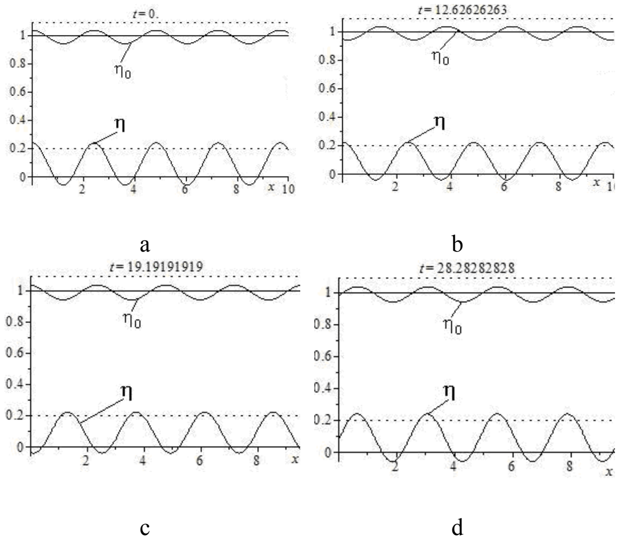
where c_1 and c_2 are given by (7) and

$$B(\omega_{1,2}) = \frac{0.5\omega_{1,2}^2}{1-\rho}\left(1-\rho-\mathrm{cth}^2(kh_1)+ \rho\left[\frac{(1-\rho)k+Tk^3-\omega_{1,2}^2\,\mathrm{cth}(kh_1)}{\rho\omega_{1,2}^2}\right]^2\right),$$

$$C(\omega_{1,2}) = \frac{0.5\omega_{1,2}^2\left(\omega_{1,2}^4-(k+T_0k^3)^2\right)}{\omega_{1,2}^2\,\mathrm{ch}(kh_2)-(k+T_0k^3)\,\mathrm{sh}(kh_2)}.$$

In Fig. 4 the dependences $\eta_2(x)$ and $\eta_{02}(x)$ are presented at different time instants for the following system parameters: $h_2 = 1$, $h_1 = 3$, $T = T_0 = 0$, $k = 1.3$, $\rho = 0.9$, $A_1 = 0.2$, $A_2^0 = 0.09$. The levels $A_1 = 0.2$ and $A_2^0 + h_2 = 1.09$ are marked by the dotted lines.

Time variations of η_{02} are faster than time variations of η_2 which means rotations of minima and maxima η_{02} and η_2 take place (Fig. 4c,d). Besides, increase or decrease in the amplitude of the wave on the contact surface takes place with time (Fig. 4a, b). For the free surface the similar picture is obtained to a lesser degree.



a b c d

Fig. 4. Second approximation of the contact surface elevation η and free surface elevation η_0 at different time instants.

4. Interaction of internal and surface waves. Substituting (6) and (8), the solutions of the evolution equations (5) and the expressions for the elevation of the contact surface and the free surface (2), one can obtain after some transformations the following expressions

$$\eta = a\cos(kx-\tilde{\omega}_1 t)+a^0a_2\cos(kx-\tilde{\omega}_2 t)+ \\ +2\alpha a^2\left(B(2\omega_1)+\Lambda(2\omega_1)\cos(2kx-2\tilde{\omega}_1 t)\right)+ \\ +2\alpha\left(a^0\right)^2\left(B(2\omega_2)+c_2\Lambda(2\omega_2)\cos(2kx-2\tilde{\omega}_2 t)\right),$$

$$\eta_0 = aa_1\cos(kx-\tilde{\omega}_1 t)+a^0\cos(kx-\tilde{\omega}_2 t)+ \\ +2\alpha a^2\left(B(2\omega_1)+c_1\Lambda(2\omega_1)\cos(2kx-2\tilde{\omega}_1 t)\right)+ \\ +2\alpha\left(a^0\right)^2\left(B(2\omega_2)+\Lambda(2\omega_2)\cos(2kx-2\tilde{\omega}_2 t)\right),$$

where $\tilde{\omega}_1 = \omega_1-\alpha^2a^2\omega_1^{-1}I$, $\tilde{\omega}_2 = \omega_2-\alpha^2(a^0)^2\omega_2^{-1}I_0$.

In Fig. 5 the dependences $\eta_2(x)$ and $\eta_{02}(x)$ are presented at different time instants for the following system parameters: $h_2 = 1$, $h_1 = 3$, $T = T_0 = 0$, $k = 1.3$, $\rho = 0.9$, $a = 0.2$, $a^0 = 0.2$, $\alpha = 0.1$. The levels $a = 0.2$ and $a^0 + h_2 = 1.2$ are marked by dotted lines.

As it is shown in Fig.5, the amplitudes of the internal and surface waves are increased and then decreased as compared to the unperturbed state $a = a_0 = 0.2$. Taking into account both pairs of the frequencies in the wave propagation, we notice that the internal waves lead to periodic sharpening and smoothing the wave crest. That is, the wave crest moves faster than the base and collapses at first (Fig. 5d), while then it is smoothed due to action of dispersion (Fig. 5c, 5d). In that way the influence of nonlinearity and dispersion are balanced.

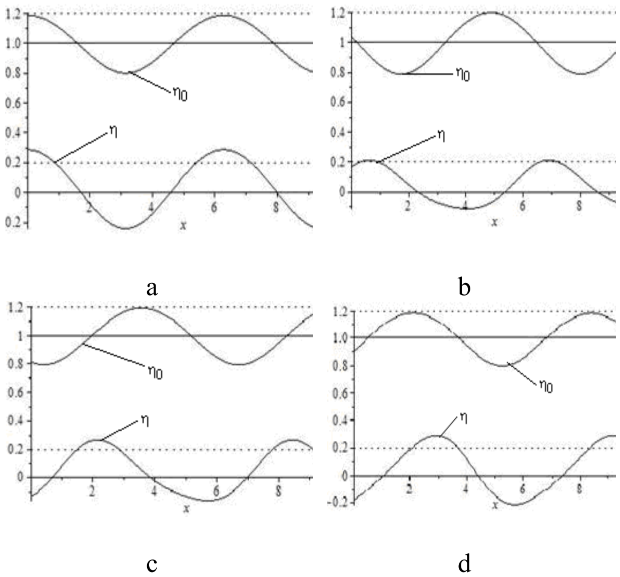


Fig. 5. Internal and surface waves at different time instants: t=0 (a), t=5 (b), t=10 (c), t=15 (d).

5. Conclusions. In the paper interaction of the internal and surface waves in the two-layer fluid with free surface has been considered. The amplitudes of the second harmonics of the elevations of the contact surface and the free surface for two pairs of frequencies of the center of the wave packet (the roots of the dispersion equation) have been investigated. The influence of nonlinearity and dispersion on propagation of internal and surface waves is revealed.

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