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A DENSITY THEOREM FOR PARTIAL SUM OF DIRICHLET SERIES

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Theorem on density of zeros inside the critical strip for the partial sum of Dirichlet series of Riemann zeta-function has been proved.

KEY WORDS: density of zeros, Dirichlet series, Riemann zeta-function, partial sum.

ТЕОРЕМА О ПЛОТНОСТИ ЧАСТИЧНЫХ СУММ РЯДА ДИРИХЛЕ

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Сформулирована и доказана теорема о плотности нулей в критической полосе для частичной суммы ряда Дирихле дзета-функции Римана.

КЛЮЧЕВЫЕ СЛОВА: плотность нулей, ряд Дирихле, дзета-функция Римана, частичная сумма.

ТЕОРЕМА ПРО ЩІЛЬНІСТЬ ЧАСТКОВИХ СУММ РЯДУ ДІРІХЛЕ

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Сформульована і доведена теорема про щільність нулів в крітічній смузі для часткових сум ряду Діріхле дзета-функції Рімана.

КЛЮЧОВІ СЛОВА: щільність нулів, ряд Діріхле, дзета-функція Рімана, часткова сума.

1. Introduction. Density theorems are the class of the theorems in the number theory which provide upper estimate of quantity of zeros of Dirichlet series and the Riemann zeta-function. The main theorem of this article is connected with an estimate of quantity of zeros of the Riemann zeta-function in the rectangle

$$|\mathrm{Im}\,\rho| \le T\ ,\ \frac{1}{2} \le \sigma \le \mathrm{Re}\,\rho \le 1\ .$$

Density theorems play an important role in studying of zero sets of zeta-function and Dirichlet series. It provides an estimate of the quantity of zeros in a given rectangle inside the critical strip. These results are very important for the number theory because they allow to understand the structure of the distribution of prime numbers. Density theorems are the powerful instrument which allows to study the behavior of the zeta-function inside the critical strip and to solve a lot of related problems. For the first time, these theorems were occurred in the articles of Hoheisel. Next important results were obtained by Selberg (see [1]), Chudakov, Bombieri (see [2]), Titchmarsh (see [3-5]) etc. Density theorems play a special role because they help to replace the Riemann hypothesis in different problems.

The Riemann zeta-function

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is used extensively in the number theory. As the function of real variable it was introduced by L. Euler in 1737. He studied its properties and found the product decomposition.

Euler product. *Let* $\mathrm{Re}\,s > 1$*.Then the following equation holds*

$$\varsigma(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where p *is a prime number.*

Then this function was investigated by Dirichlet. For the first time, the zeta-function was connected with the distribution law of prime numbers by Chebyshev. But the most fundamental properties of the zeta function have been discovered by Riemann in 1859. In his paper Riemann considered it as the function of complex variable, found formula for determination the quantity of prime numbers under the given one, suggested his popular hypothesis about zeros of the zeta-function.

Riemann hypothesis. *All zeros of the Riemann zeta-function have real part* $\mathrm{Re}(s) = 1/2$.

The problem of behavior of Riemann zeta-function in the critical strip, especially the problem of its zero sets, is one of the most difficult and most interesting problems of mathematical analysis. Solution of this problem is connected with the solution of central problem of analytical number theory as the distribution of prime numbers in a sequence.

In this article the theorem on the density of zeros inside the strip $\mathrm{Re}\, s \in (0,1)$ for the partial sum of Dirichlet series has been obtained.

2. Formulation of the main result.

Definition. The partial sum of Dirichlet series of the Riemann zeta-function is the function of the form

$$\varsigma_0(s) = \sum_{n=1}^{N} \frac{1}{n^s}.$$

Definition. *Let* ρ *be a zero of the function* $\varsigma_0(s)$, *and let* $0 \le \sigma \le 1$, $T \ge 2$. *The function* $N_0(\sigma, T)$ *is the function of the form*

$$N_0(\sigma, T) = \sum_{\substack{|\mathrm{Im}\,\rho| \le T \\ \sigma \le \mathrm{Re}\,\rho \le 1}} 1;$$

i.e. $N_0(\sigma, T)$ *is the quantity of zeros of the function* $\varsigma_0(s)$ *in the rectangle* $|\mathrm{Im}\,\rho| \le T$, $\sigma \le \mathrm{Re}\,\rho \le 1$.

Theorem 1. *If* $0.5 \le \sigma \le 1$, $T \le N$ *then the following estimate holds*

$$N_0(\sigma, T) \le cT^{4\sigma(1-\sigma)} (\ln N)^2 (\ln T)^8,$$

where $C > 0$ *doesn't depend on* T *and* N.

This theorem is similar to the density theorem for the zeta function (see [6]). But the function $\varsigma_0(s)$ has other structure and different properties. So proof of this theorem required new ideas.

3. Auxiliary statements. Investigation of the function $\varsigma_0(s)$ gives some interesting results.

The location of zeros of the function $\varsigma_0(s)$ has been determined. It is totally different from the location of zeros of the zeta function. The Riemann zeta-function has trivial and non-trivial zeros. The trivial zeros are $s = -2, -4, -6...$, while the non-trivial zeros are located inside the critical strip $\mathrm{Re}\, s \in (0,1)$. The location of zeros of the function $\varsigma_0(s)$ is described in the following lemma.

Lemma 1. *All the zeros of the function* $\varsigma_0(s)$ *are contained in the strip*

$$-\frac{\ln N}{\ln N - \ln(N-1)} < \mathrm{Re}\, s < 2 - \varepsilon,$$

where ε *is a sufficiently small positive number which doesn't depend on* N.

The zeta-function isn't an entire function but the function $\varsigma_0(s)$ is. Therefore $\varsigma_0(s)$ has interesting properties and one can to find the following product decomposition.

Theorem 2. $\varsigma_0(s)$ *is an entire first-order function and the following representation is true*

$$\varsigma_0(s) = Ne^{Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\rho_n}\right),$$

where $\rho_n = \beta_n + i\gamma_n$ *are zeros of the function* $\varsigma_0(s)$, $B \ge -\ln N$.

Taking into account this property it may be find the following formula which helps to prove the main theorem.

Corollary. *The following formula is true*

$$\frac{\varsigma_0'(s)}{\varsigma_0(s)} = B + \sum_{n=1}^{\infty} \frac{1}{s - \rho_n}.$$

It was necessary to get the next result as an auxiliary statements.

Theorem 3. *Let* $\rho_n = \beta_n + i\gamma_n$ *be the zeros of the function* $\varsigma_0(s)$, $T \ge 2$. *Then the following estimate is valid*

$$\sum_{n=1}^{\infty} \frac{1}{(2-\beta_n)^2 + (T-\gamma_n)^2} \le c_1 \ln N.$$

The following result for the function $\varsigma_0(s)$ is identical with the correspondent result for the zeta-function.

Corollary. The *quantity of zeros* ρ_n *of the function* $\varsigma_0(s)$ *in the domain* $0 < \beta_n < 1$, $T \le |\gamma_n| \le T+1$ *doesn't exceed* $c_2 \ln N$.

4. Conclusion. It is produce a density theorem for partial sum of Riemann zeta-function in the article (Theorem 1). This theorem provides an estimate of quantity of zeros in the rectangle inside the critical strip.

Proving the theorem on the function $\varsigma_0(s)$ has been investigated. It has been obtained some interesting results as the location of the zeros of the function, the product decomposition etc.

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