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FULL AVERAGING OF SET INTEGRODIFFERENTIAL EQUATIONS ON A FINITE INTERVAL

Komleva T.A.

Odessa State Academy of Civil Engineering and Architecture, Ukraine

In the paper the substantiation of the method of full averaging for the set of integrodifferential equations with small parameter is considered. Thereby, a circle of systems allowing application of Krylov-Bogolyubov method of averaging is expanded.

KEY WORDS: integrodifferential equations, full averaging, Krylov-Bogolyubov method.

ПОЛНОЕ ОСРЕДНЕНИЕ СИСТЕМЫ ИНТЕГРОДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ НА КОНЕЧНОМ ИНТЕРВАЛЕ

Комлева Т.А.

В работе содержится обоснование метода полного осреднения системы интегродифференциальных уравнений с малым параметром. Таким образом, проведено расширение круга систем, допускающих использование метода осреднения Крылова-Боголюбова.

КЛЮЧЕВЫЕ СЛОВА: интегродифференциальные уравнения, полное осреднение, метод Крылова-Боголюбова.

ПОВНЕ ОСЕРЕДНЕННЯ СИСТЕМИ ІНТЕГРОДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ НА СКІНЧЕНОМУ ІНТЕРВАЛІ

Комлєва Т.А.

В роботі міститься обґрунтування методу повного осереднення системи інтегродиференціальних рівнянь з малим параметром. Таким чином, проведено розширення кола систем, які допускають використання методу усереднення Крилова-Боголюбова.

КЛЮЧОВІ СЛОВА: інтегродиференціальні рівняння, повне осереднення, метод Крилова-Боголюбова.

1. Introduction. As is generally known the absence of exact universal research methods for many important problems of analytical dynamics has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. It became possible due to the works of N.M. Krylov, N.N. Bogolyubov, Yu.A. Mitropolskij, V.M. Volosov, N.N. Moiseev, etc. (see [1–7]).

In recent years the development of the calculus in metric spaces has attracted some attention [5–9]. Earlier, F.S. de Blasi, F. Iervolino [10] started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline: properties of solutions [11–14], the impulse equations [5,6,22], control systems [23–25] and asymptotic methods [5–7,25]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions [5,18,20]. Also, one can employ SDEs in the investigation of fuzzy differential equations [6,9,15,17,18].

In this paper the substantiation of the method of full averaging for the set integrodifferential equations with small parameter is considered. Thereby, a circle of systems allowing application of Krylov-Bogolyubov method of averaging is expanded.

2. Preliminaries. Let $\mathrm{conv}(R^n)$ be a set of all nonempty convex and compact subsets from the space R^n, $h(A,B)=\min_{r\geq 0}\{\, S_r(A)\supset B,\ S_r(B)\supset A \,\}$ be the Hausdorff distance between sets A and B, $S_r(A)$ is r-neighborhood of set A.

Let A, B, C be in $\mathrm{conv}(R^n)$. The set C is the Hukuhara difference of A and B, if $B+C=A$, i.e.

$$C = A \overset{H}{—} B.$$

From Radstrom's Cancellation Lemma [26], it follows that if this difference exists, then it is unique.

Definition [27]. A mapping $X:[0,T] \to \text{conv}(R^n)$ is differentiable in the sense of Hukuhara at $t \in [0,T]$ if for some $\delta > 0$ the Hukuhara differences $X(t+\Delta) \overset{H}{—} X(t)$, $X(t) \overset{H}{—} X(t-\Delta)$ exists in $\text{conv}(R^n)$ for all $0 < \Delta < \delta$ and there exists an $DX(t) \in \text{conv}(R^n)$ such that

$$\lim_{\Delta \to 0_+} h\left(\Delta^{-1}\left(X(t+\Delta) \overset{H}{—} X(t)\right), DX(t)\right) = 0$$

and

$$\lim_{\Delta \to 0_+} h\left(\Delta^{-1}\left(X(t) \overset{H}{—} X(t-\Delta)\right), DX(t)\right) = 0.$$

Here $DX(t)$ is called the Hukuhara derivative of $X(t)$ at t.

3. The scheme of full average. Consider the Cauchy problem with small parameter

$$DX = \varepsilon F\left(t, X, \int_0^t \Phi(t, s, X(s))ds\right), \quad X(0) = X_0, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $t, s \in R_+$, $F: R_+ \times \text{conv}(R^n) \times \text{conv}(R^m) \to \text{conv}(R^n)$ is a set mapping, $\Phi: R_+ \times R_+ \times \text{conv}(R^n) \to \text{comp}(R^m)$ is a set mapping. Here the integral is understood in the sense of [27] (the integral exists for example if $X(\cdot)$ is measurable and the real mapping $t \to h(X(t), \{0\})$ is integrable on $I \subset R_+$), $X_0 \in \text{conv}(R^n)$.

Definition. A mapping $X:[0,T] \to \text{conv}(R^n)$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation $X(t) = X_0 + \varepsilon \int_0^t F\left(\tau, X(\tau), \int_0^\tau \Phi(\tau, s, X(s))ds\right)d\tau$ for all $t \in [0,T]$.

Let us associate with equation (1) the following full averaged integrodifferential equation

$$DY = \varepsilon \overline{F}(Y), \qquad Y(0) = X_0, \tag{2}$$

where

$$\lim_{T \to \infty} h\left(\frac{1}{T}\int_0^T F_1(t, X)dt, \overline{F}(X)\right) = 0, \tag{3}$$

$$F_1(t, X) \equiv F(t, X, \Psi(t, X)),$$
$$\Psi(t, X) \equiv \int_0^t \Phi(t, s, X)ds. \tag{4}$$

Suppose that limit (3) exists.

Theorem. *Let in the domain* $Q = \{(t, X) \mid t \geq 0, X \in G \in \text{conv}(R^n)\}$ *the following hold:*

1) $F(t, X, Z)$ *is continuous in* $(t, X, Z) \in R_+ \times G \times \text{conv}(R^m)$*;*

2) $\Phi(t, s, X)$ *is continuous in* $(t, s, X) \in R_+ \times R_+ \times G$*;*

3) *there exist continuous function* $\mu(t, s)$ *and constant* λ *such that*

$$h(F(t, X_1, Z_1), F(t, X_2, Z_2)) \leq \lambda[h(X_1, X_2) + h(Z_1, Z_2)],$$
$$h(\Phi(t, s, X_1), \Phi(t, s, X_2)) \leq \mu(t, s)h(X_1, X_2),$$

for any $X_1, X_2 \in G$, $Z_1, Z_2 \in \text{conv}(R^m)$*;*

4) $\lim_{t \to \infty} \overline{\mu}_0(t) = 0$, *where* $\overline{\mu}_0(t) \equiv \frac{1}{t}\int_0^t \mu_0(\tau)d\tau$, $\mu_0(\tau) \equiv \int_0^\tau \mu(\tau, s)ds$*;*

5) *there exist constants* ν, M *such that* $h(\overline{F}(X_1), \overline{F}(X_2)) \leq \nu h(X_1, X_2)$, $h(\overline{F}(X), \{0\}) \leq M$ *for any* $X, X_1, X_2 \in G$*;*

6) *limits (3) exists uniformly in* $X \in G$*;*

7) *for any* $X_0 \in G' \subset G$ *and* $t \geq 0$ *the solution of equation (2) together with a* σ*-neighborhood belongs to the domain* G.

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon^0(\eta, L) \in (0, \sigma]$ *such that for all* $\varepsilon \in (0, \varepsilon^0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the following statement fulfill:*

$$h(X(t), Y(t)) < \eta, \tag{5}$$

where $X(t)$, $Y(t)$ *are the solutions of the initial and the averaged equations.*

Proof. Since

$$X(t) = X_0 + \varepsilon \int_0^t F\left(\tau, X(\tau), \int_0^\tau \Phi(\tau, s, X(s))ds\right)d\tau,$$

$$Y(t) = X_0 + \varepsilon \int_0^t \overline{F}(Y(\tau))d\tau,$$

we have $h(X(t), Y(t)) \leq$

$$\le \varepsilon h\left(\int_0^t F\left(\tau, X(\tau), \int_0^\tau \Phi(\tau,s,X(s))ds\right)d\tau,\right.$$

$$\left.\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(s))ds\right)d\tau\right)+$$

$$+\varepsilon h\left(\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(s))ds\right)d\tau,\right.$$

$$\left.\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau\right)+$$

$$+\varepsilon h\left(\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau, \int_0^t \overline{F}(Y(\tau))d\tau\right).$$

Hence $h(X(t),Y(t)) \le$

$$\le \varepsilon\int_0^t \lambda\left[\begin{array}{l} h(X(\tau),Y(\tau))+ \\ h\left(\int_0^\tau \Phi(\tau,s,X(s))ds, \int_0^\tau \Phi(\tau,s,Y(s))ds\right)\end{array}\right]d\tau+$$

$$+\varepsilon\int_0^t \lambda h\left(\int_0^\tau \Phi(\tau,s,Y(s))ds, \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau+$$

$$+\varepsilon h\left(\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau, \int_0^t \overline{F}(Y(\tau))d\tau\right).$$

Then we obtain

$$\begin{aligned} h(X(t),Y(t)) &\le \varepsilon\lambda\int_0^t h(X(\tau),Y(\tau))d\tau+ \\ &+\varepsilon\lambda\int_0^t\int_0^\tau \mu(\tau,s)h(X(s),Y(s))dsd\tau+ \\ &+\varepsilon\lambda\int_0^t\int_0^\tau \mu(\tau,s)h(Y(s),Y(\tau))dsd\tau+ \\ &+\varepsilon h\left(\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau, \int_0^t \overline{F}(Y(\tau))d\tau\right). \end{aligned} \qquad (6)$$

First let us estimate the last summand in (6). Divide the interval $[0, L\varepsilon^{-1}]$ into partial intervals by the points $t_i = \frac{iL}{m\varepsilon}$, $i = 0,...,m$, $m \in N$. Then

$$\varepsilon h\left(\int_0^t F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau))ds\right)d\tau, \int_0^t \overline{F}(Y(\tau))d\tau\right) \le$$

$$\Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 = \varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(F\left(\tau, Y(\tau), \int_0^\tau \Phi(\tau,s,Y(\tau)ds)\right),\right.$$

$$\left.F\left(\tau, Y(t_i), \int_0^\tau \Phi(\tau,s,Y(t_i))ds\right)\right)d\tau,$$

$$\Sigma_2 = \varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(\overline{F}(Y(t_i)), \overline{F}(Y(\tau))\right)d\tau,$$

$$\Sigma_3 = \varepsilon\sum_{i=0}^{m-1} h\left(\int_{t_i}^{t_{i+1}} F\left(\tau, Y(t_i), \int_0^\tau \Phi(\tau,s,Y(t_i))ds\right)d\tau,\right.$$

$$\left.\int_{t_i}^{t_{i+1}} \overline{F}(Y(t_i))d\tau\right).$$

As $h(Y(\tau),Y(t_i)) \le \varepsilon M|\tau - t_i|$ for all $\tau \in [t_i, t_{i+1}]$, $i = \overline{0, m-1}$; then

$$\Sigma_1 \le$$

$$\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\left[\lambda h(Y(\tau),Y(t_i)) + \lambda\int_0^\tau \mu(\tau,s)h(Y(\tau),Y(t_i))ds\right]d\tau \le$$

$$\le \varepsilon^2\lambda M\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\left[(\tau - t_i) + \mu_0(\tau)(\tau - t_i)\right]d\tau$$

$$\le \lambda\frac{ML^2}{2m} + \varepsilon\lambda\frac{ML}{m}\int_0^{L/\varepsilon}\mu_0(\tau)d\tau,$$

$$\Sigma_2 \le \varepsilon\nu\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h(Y(\tau),Y(t_i))d\tau \le$$

$$\le \varepsilon^2\nu M\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}(\tau - t_i)d\tau \le \frac{\nu ML^2}{2m}.$$

Hence

$$\Sigma_1 + \Sigma_2 \le \frac{ML^2}{2m}(\lambda + \nu) + \frac{\lambda ML}{m}\beta(\varepsilon) \equiv \alpha(m,\varepsilon),$$

where $\beta(\varepsilon) \equiv \sup_{0\le\tau\le L}\tau\overline{\mu}_0\left(\frac{\tau}{\varepsilon}\right) \ge \varepsilon t\overline{\mu}_0(t) = \varepsilon\int_0^t \mu_0(\tau)d\tau$.

Obviously, $\lim_{m\to\infty}\alpha(m,\varepsilon) = 0$, and $\lim_{\varepsilon\to 0}\beta(\varepsilon) = 0$.

Now, let us estimate Σ_3. From condition 6) of the theorem it follows that there exists an increasing function $\theta(t)$ such that

1) $\lim_{t\to\infty}\theta(t) = 0$;

2) $h\left(\int_0^t F_1(\tau,X,)d\tau, \int_0^t \overline{F}(X)d\tau\right) \le t\theta(t)$.

Then

$$\Sigma_3 \le \varepsilon \sum_{i=0}^{m-1} h\left(\int_0^{t_{i+1}} F\left(\tau, Y(t_i), \int_0^{\tau} \Phi(\tau, s, Y(t_i))ds\right)d\tau, \int_0^{t_{i+1}} \overline{F}(Y(t_i))d\tau \right) + \varepsilon \sum_{i=0}^{m-1} h\left(\int_0^{t_i} F\left(\tau, Y(t_i), \int_0^{\tau} \Phi(\tau, s, Y(t_i))ds\right)d\tau, \int_0^{t_i} \overline{F}(Y(t_i))d\tau \right) \le 2m\psi(\varepsilon).$$

where $\psi(\varepsilon) = \sup\limits_{\tau\in[0,L]} \left(\tau\theta\left(\frac{\tau}{\varepsilon}\right)\right)$, $\tau = \varepsilon\, t$.

By (6) we have

$$h(X(t), Y(t)) \le \varepsilon\lambda \int_0^t h(X(\tau), Y(\tau))d\tau +$$
$$+\varepsilon\lambda \int_0^t \int_0^{\tau} \mu(\tau, s) h(X(s), Y(s))dsd\tau +$$
$$+\varepsilon\lambda \int_0^t \int_0^{\tau} \mu(\tau, s) h(Y(s), Y(\tau))ds\, d\tau + \alpha(m, \varepsilon) +$$
$$+2m\psi(\varepsilon) \le \varepsilon\lambda \int_0^t h(X(\tau), Y(\tau))d\tau +$$
$$+\int_0^t \int_0^{\tau} \mu(\tau, s) h(X(s), Y(s))dsd\tau +$$
$$+\lambda ML\beta(\varepsilon) + \alpha(m, \varepsilon) + 2m\psi(\varepsilon).$$

Using Gronwall-Bellman's inequality, we obtain

$$h(X(t), Y(t)) \le \left(\alpha(m, \varepsilon) + \lambda ML\beta(\varepsilon) + 2m\psi(\varepsilon) \right) e^{\lambda L + \lambda\beta(\varepsilon)}.$$

If m is fixed then for any $\eta > 0$ exists $\varepsilon_0 > 0$ such that the following estimate is true for $0 < \varepsilon < \varepsilon_0$:

$$\alpha(m, \varepsilon) + \lambda ML\beta(\varepsilon) + 2m\psi(\varepsilon) < \min\{\eta, \sigma\} e^{-\lambda(1+L)},$$
$$\beta(\varepsilon) < 1.$$

Hence we get $h(X(t), Y(t)) \le \eta$ for all $t \in [0, L\varepsilon^{-1}]$. This concludes the proof.

4. Conclusions. Here we used the approach of Hukuhara at definition of the derivative which has essential shortages. However the given approach is well investigated by many authors. Also in the literature exist other approaches to definition of the derivative [6,7,28–30], but they also have the shortages. It is easily possible to show that this result will be true for some other cases with little changes.

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