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# TOWARD THE DIRICHLET PROBLEM IN FINITELY CONNECTED DOMAINS

***Kovtonyuk D.A., Petkov I.V., Ryazanov V.I., Salimov R.R.***

Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine, Donetsk

A survey of the recent theorems on existence of regular, pseudo-regular and multi-valued solutions of the Dirichlet problem to the Beltrami equations with degeneration in arbitrary finitely connected domains bounded by mutually disjoint Jordan curves is given.

*KEY WORDS*: Beltrami equations, Dirichlet problem, theorem on existence.

## К ЗАДАЧЕ ДИРИХЛЕ В КОНЕЧНОСВЯЗАННЫХ ОБЛАСТЯХ

***Ковтонюк Д.А., Петков И.В., Рязанов В.И., Салимов Р.Р.***

В работе приведен обзор теорем существования регулярных, псевдорегулярных и многозначных решений задачи Дирихле для вырожденных уравнений Бельтрами в произвольных конечносвязанных областях ограниченных взаимно непересекающимися Жордановыми кривыми.

*КЛЮЧЕВЫЕ СЛОВА*: уравнения Бельтрами, задача Дирихле, теорема существования.

## ДО ЗАДАЧІ ДІРІХЛЕ У СКІНЧЕНО ЗВ'ЯЗАНИХ ОБЛАСТЯХ

***Ковтонюк Д.А., Петков І.В., Рязанов В.І., Салимов Р.Р.***

У роботі наведено огляд теорем існування регулярних, псевдорегулярних і багатозначних рішень задачі Діріхле для вироджених рівнянь Бельтрамі в довільних конечносвязанних областях обмежених взаємно непересічними жорданова кривими.

*КЛЮЧОВІ СЛОВА*: рівняння Бельтрамі, задача Діріхле, теорема існування.

**1. Introduction.** Here we give a survey of our recent results in the Dirichlet problem for the Beltrami equations with degeneration published in the series of papers [1]–[4]. Namely, we formulate a number of criteria for existence of regular solutions to this problem in arbitrary Jordan domains and pseudo-regular and multi-valued solutions in arbitrary finitely connected domains bounded by mutually disjoint Jordan curves.

So, let $D$ be a domain in the complex plane $C$, i.e., a connected open subset of $C$, and let $\mu : D \to C$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in $D$. A Beltrami equation is an equation of the form

$$f_{\bar z} = \mu(z) f_z \tag{1.1}$$

where $f_{\bar z} = \bar\partial f = \frac{1}{2}\left(f_x + i f_y\right)$, $f_z = \partial f = \frac{1}{2}\left(f_x - i f_y\right)$, $z = x + iy$ and $f_x$ and $f_y$ are partial derivatives of $f$ in $x$ and $y$, correspondingly. The function $\mu$ is called the complex coefficient and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the *dilatation quotient* or simply the *dilatation* of the equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

Recall that every analytic function $f$ in a domain $D \subset C$ satisfies the simplest Beltrami equation

$$f_{\bar z} = 0 \tag{1.3}$$

with $\mu(z) \equiv 0$. If an analytic function $f$ given in the unit disk $D$ is continuous in its closure, then by the Schwarz formula

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi i} \int_{|\zeta|=1} \operatorname{Re} f(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}, \tag{1.4}$$

see, e.g., Section 8, Chapter III, Part 3 in [5]. Thus, the analytic function f in the unit disk $D$ is determined, up to a purely imaginary additive constant $ic$, $c = \operatorname{Im} f(0)$, by its real part $\phi(\zeta) = \operatorname{Re} f(\zeta)$ on the boundary of $D$.

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Hence the *Dirichlet problem* for the Beltrami equation (1.1) in a domain $D \subset C$ is the problem on the existence of a continuous function $f : D \to C$ having partial derivatives of the first order a.e., satisfying (1.1) a.e. and such that

$$\lim_{z\to\zeta} \operatorname{Re} f(z) = \phi(\zeta) \qquad \forall \zeta \in \partial D \qquad (1.5)$$

for a prescribed continuous function $\phi : \partial D \to R$. It is obvious that if $f$ is a solution of this problem, then the function $F(z) = f(z) + ic$, $c \in R$, is so.

The existence of homeomorphism $W^{1,1}_{loc}$ solutions was recently established for many degenerate Beltrami equations, see, e.g., related references in the recent monographs [6] and [7] and in the surveys [8] and [9]. Boundary value problems for the Beltrami equations are due to the well-known Riemann dissertation in the case of $\mu(z) \equiv 0$ and to the papers of Hilbert (1904, 1924) and Poincare (1910) for the corresponding Cauchy–Riemann system. The Dirichlet problem for uniformly elliptic systems was studied long ago, see, e.g., [10] and [11]. The Dirichlet problem for degenerate Beltrami equations in the unit disk was studied in [12].

Throughout this paper,

$B(z_0, r) = \{z \in C : |z - z_0| < r\}$, $D_0 = B(0,1)$,

$R(z_0, r_1, r_2) = \{z \in C : r_1 < |z - z_0| < r_2\}$.

$S(z_0, r) = \{z \in C : |z - z_0| = r\}$, $S(r) = S(0, r)$,

**2. BMO and FMO functions**. The well-known class BMO was introduced by John and Nirenberg in the paper [13] and soon became an important concept in harmonic analysis, partial differential equations and related areas; see, e.g., [14] and [15]. Recall that a real-valued function $u$ in a domain $D$ in $C$ is said to be of *bounded mean oscillation in* $D$, abbr. $u \in BMO(D)$, if $u \in L^1_{loc}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| \, dm(z) < \infty, \qquad (2.1)$$

where the supremum is taken over all discs $B$ in $D$, $dm(z)$ corresponds to the Lebesgue measure in $C$ and

$$u_B = \frac{1}{|B|} \int_B u(z) \, dm(z).$$

We write $u \in BMO_{loc}(D)$ if $u \in BMO(U)$ for every relatively compact subdomain $U$ of $D$ (we also write BMO or $BMO_{loc}$ if it is clear from the context what $D$ is).

Following the paper [16], see also [6] and [7], we say that a function $\phi : D \to R$ has *finite mean oscillation at a point* $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{|B(z_0,\varepsilon)|} \int_{B(z_0,\varepsilon)} |\phi(z) - \tilde{\phi}(z_0)| \, dm(z) < \infty, \quad (2.2)$$

where

$$\tilde{\phi}(z_0) = \frac{1}{|B(z_0,\varepsilon)|} \int_{B(z_0,\varepsilon)} \phi(z) \, dm(z) \qquad (2.3)$$

is the mean value of the function $\phi(z)$ over the disk $B(z_0, \varepsilon)$. Note that the condition (2.2) includes the assumption that $\phi$ is integrable in some neighborhood of the point $z_0$. We say also that a function $\phi : D \to R$ is of *finite mean oscillation in* $D$, abbr. $\phi \in FMO(D)$ or simply $\phi \in FMO$, if $\phi \in FMO(z_0)$ for all points $z_0 \in D$. We write $\phi \in FMO(\overline{D})$ if $\phi$ is given in a domain $G$ in $C$ such that $\overline{D} \subset G$ and $\phi \in FMO(z_0)$ for all $z_0 \in \overline{D}$.

The following statement is obvious by the triangle inequality.

**Proposition 2.1.** *If, for a collection of numbers* $\phi_\varepsilon \in R$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{|B(z_0,\varepsilon)|} \int_{B(z_0,\varepsilon)} |\phi(z) - \phi_\varepsilon| \, dm(z) < \infty, \quad (2.4)$$

*then* $\phi$ *is of finite mean oscillation at* $z_0$.

In particular choosing in Proposition 2.1, $\phi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we obtain the following statement.

**Corollary 2.1.** *If, for a point* $z_0 \in D$,

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{|B(z_0,\varepsilon)|} \int_{B(z_0,\varepsilon)} |\phi(z)| \, dm(z) < \infty, \quad (2.5)$$

*then* $\phi$ *is of finite mean oscillation at* $z_0$.

Recall that a point $z_0 \in D$ is called a *Lebesgue point* of a function $\phi : D \to R$ if $\phi$ is integrable in a neighborhood of $z_0$ and

$$\lim_{\varepsilon\to 0} \frac{1}{|B(z_0,\varepsilon)|} \int_{B(z_0,\varepsilon)} |\phi(z) - \phi(z_0)| \, dm(z) = 0. \quad (2.6)$$

It is known that, almost every point in $D$ is a Lebesgue point for every function $\phi \in L^1(D)$. Thus, we have by Proposition 2.1 the following corollary showing that the FMO condition is very natural.

**Corollary 2.2.** *Every locally integrable function* $\phi : D \to R$ *has a finite mean oscillation at almost every point in* $D$.

**Remark 2.1.** Note that the function $\phi(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\Delta$, see, e.g., [14], p. 5, and hence also to FMO. However, $\tilde{\phi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (2.5) is only sufficient but not necessary for a function $\phi$ to be of finite mean oscillation at $z_0$. Clearly, $BMO(D) \subset BMO_{loc}(D) \subset FMO(D)$ and as well-known $BMO_{loc} \subset L^p_{loc}$ for all $p \in [1, \infty)$, see, e.g., [14]. However, FMO is not a subclass of $L^p_{loc}$ for any $p > 1$ but only of $L^1_{loc}$, see examples in [7], p. 211.

Thus, the class FMO is essentially wider than $BMO_{loc}$.

**3. On regular solutions for the Dirichlet problem in Jordan domains.** If $\phi(\zeta)\neq \text{const}$, then the *regular solution* of such a problem is a continuous, discrete and open mapping $f:D\to C$ of the Sobolev class $W^{1,1}_{loc}$ with its Jacobian $J_f(z)=|f_z|^2-|f_{\bar z}|^2\neq 0$ a.e. satisfying (1.1) a.e. and the condition (1.5). Recall that a mapping $f:D\to C$ is called *discrete* if the preimage $f^{-1}(y)$ consists of isolated points for every $y\in C$, and *open* if $f$ maps every open set $U\subseteq D$ onto an open set in $C$. The regular solution of the Dirichlet problem (1.5) with $\phi(\zeta)\equiv c$, $\zeta\in\partial D$, for the Beltrami equation (1.1) is the function $f(z)\equiv c$, $z\in D$.

**Theorem 3.1.** *Let $D$ be a Jordan domain and $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e. such that $K_\mu(z)\le Q(z)$ a.e. in $D$ for a function $Q:C\to[0,\infty]$ in $FMO(\bar D)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.5) for each continuous function $\phi:\partial D\to R$.*

**Corollary 3.1.** *In particular, the conclusion of Theorem 3.1 holds if every point $z_0\in\bar D$ is the Lebesgue point of a locally integrable function $Q:C\to[0,\infty]$ such that $K_\mu(z)\le Q(z)$ a.e. in $D$.*

Further we assume that $K_\mu$ is extended by zero outside of $D$.

**Corollary 3.2.** *Let $D$ be a Jordan domain and $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e. such that*

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{|B(z_0,\varepsilon)|}\int_{B(z_0,\varepsilon)}K_\mu(z)\,dm(z)<\infty \qquad (3.1)$$

$$\forall z_0\in\bar D$$

*Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.5) for each continuous function $\phi:\partial D\to R$.*

**Theorem 3.2.** *Let $D$ be a Jordan domain in $C$ and $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e. If $K_\mu\in L^1_{loc}(D)$ and satisfies the condition*

$$\int_0^{\delta(z_0)}\frac{dr}{\|K_\mu\|_1(z_0,r)}=\infty \quad \forall z_0\in\bar D \qquad (3.2)$$

*for some $\delta(z_0)\in(0,d(z_0))$ where $d(z_0)=\sup_{z\in D}|z-z_0|$ and*

$$\|K_\mu\|_1(z_0,r)=\int_{D\cap S(z_0,r)}K_\mu(z)\,|dz|, \qquad (3.3)$$

*at each point $z_0\in\bar D$, then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.5) for each continuous function $\phi:\partial D\to R$.*

**Corollary 3.3.** *Let $D$ be a Jordan domain and $\mu:D\to C$ be a measurable function such that*

$$k_{z_0}(\varepsilon)=O\left(\log\frac{1}{\varepsilon}\right) \qquad \forall z_0\in\bar D \qquad (3.4)$$

*as $\varepsilon\to 0$, where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu(z)$ over $S(z_0,\varepsilon)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.5) for each continuous function $\phi:\partial D\to R$.*

**Remark 3.1.** In particular, the conclusion of Corollary 3.3 holds if

$$K_\mu(z)=O\left(\log\frac{1}{|z-z_0|}\right) \text{ as } z\to z_0 \quad \forall z_0\in\bar D \qquad (3.5)$$

**Theorem 3.3.** *Let $D$ be a Jordan domain and $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e. such that*

$$\int_D \Phi(K_\mu(z))\,dm(z)<\infty \qquad \textbf{(3.6)}$$

*for a convex non-decreasing function $\Phi:[0,\infty]\to[0,\infty]$. If*

$$\int_\delta^\infty\frac{d\tau}{\tau\Phi^{-1}(\tau)}=\infty \qquad \textbf{(3.7)}$$

*for some $\delta>\Phi(0)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.5) for each continuous function $\phi:\partial D\to R$.*

**Remark 3.2.** By the Stoilow theorem, see, e.g., [17], a regular solution $f$ of the Dirichlet problem (1.5) for the Beltrami equation (1.1) with $K_\mu\in L^1_{loc}(D)$ can be represented in the form $f=h\circ F$ where $h$ is an analytic function and $F$ is a homeomorphic regular solution of (1.1) in the class $W^{1,1}_{loc}$. Thus, by Theorem 5.1 in [18] the condition (3.7) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1.5) for an arbitrary Beltrami equation (1.1) with the integral constraints (3.6) for any non-constant continuous function $\phi:\partial D\to R$.

Setting $H(t)=\log\Phi(t)$, note that by Theorem 2.1 in [19] the condition (3.7) is equivalent to each of the conditions

$$\int_\Delta^\infty H'(t)\frac{dt}{t}=\infty, \qquad (3.8)$$

$$\int_\Delta^\infty\frac{dH(t)}{t}=\infty, \qquad (3.9)$$

and (3.9) implies

$$\int_{\Delta}^{\infty} H(t)\frac{dt}{t^2} = \infty \qquad (3.10)$$

for some $\Delta > 0$, and

$$\int_{0}^{\delta} H\left(\frac{1}{t}\right)dt = \infty \qquad (3.11)$$

for some $\delta > 0$,

$$\int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \qquad (3.12)$$

for some $\Delta_* > H(+0)$. Here, the integral in (3.9) is understood as the Lebesgue– Stieltjes integral and the integrals in (3.7) and (3.10)–(3.12) as the ordinary Lebesgue integrals. Moreover, if the function $\Phi : [0,\infty] \to [0,\infty]$ is non-decreasing and convex, then all conditions (3.7)–(3.12) are equivalent each to other.

**Corollary 3.4.** *In particular, the conclusion of Theorem 3.3 holds if, for some* $\alpha > 0$,

$$\int_{D} e^{\alpha K_\mu(z)} dm(z) < \infty . \qquad (3.13)$$

4. On pseudoregular and multi-valued solutions in finitely connected domains. It was first noted by Bojarski, see, e.g., section 6 of Chapter 4 in [11], in the case of multiply connected domains the Dirichlet problem for the Beltrami equation, generally speaking, has no solutions in the class of continuous (simply-valued) functions. Hence it is arose the question: whether the existence of solutions of the Dirichlet problem can be obtained for the case in a wider class ? It is turned out to be that this is possible in the class of functions having a certain number of poles at prescribed points in *D*. More precisely, for a continuous function $\phi(\zeta) \neq \text{const}$, a *pseudoregular solution* of the problem is a continuous (in $\overline{C} = C \cup \{\infty\}$) discrete open mapping $f : D \to \overline{C}$ in the class $W^{1,1}_{loc}$ (outside of these poles) with the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0$ a.e. satisfying (1.1) a.e. and the condition (1.5). Furthermore, one can choose in the pseudoregular solution just *n* prescribed poles where *n* is equal to the number of components of the boundary of the domain *D*.

In finitely connected domains D in C, in addition to pseudoregular solutions, the Dirichlet problem (1.5) for the Beltrami equation (1.1) admits multi-valued solutions in the spirit of the theory of multi-valued analytic functions. We say that a continuous discrete open mapping $f : B(z_0,\varepsilon_0) \to C$, where $B(z_0,\varepsilon_0) \subseteq D$, is a *local regular solution of the equation* (1.1) if $f \in W^{1,1}_{loc}$, $J_f(z) \neq 0$ and f satisfies (1.1) a.e. in $B(z_0,\varepsilon_0)$.

The local regular solutions $f : B(z_0,\varepsilon_0) \to C$ and $f_* : B(z_*,\varepsilon_*) \to C$ of the equation (1.1) will be called extension of each to other if there is a finite chain of such solutions $f_i : B(z_i,\varepsilon_i) \to C$, $i = 1,\ldots,m$, that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) = f_{i+1}(z)$ for $z \in E_i := B(z_i,\varepsilon_i) \cap B(z_{i+1},\varepsilon_{i+1}) \neq \varnothing$, $i = 1,\ldots,m-1$. A collection of local regular solutions $f_j : B(z_j,\varepsilon_j) \to C$, $j \in J$, is called by us a *multi-valued solution* of the equation (1.1) in D if the disks $B(z_j,\varepsilon_j)$ cover the whole domain D and $f_j$ are extensions of each to other through the collection. A multi-valued solution of the equation (1.1) is called by us a *multi-valued solution of the Dirichlet problem* (1.5) if $u(z) = \mathrm{Re}\, f(z) = \mathrm{Re}\, f_j(z)$, $z \in B(z_j,\varepsilon_j)$, $j \in J$, is a simply-valued function in D satisfying the condition $\lim_{z\to\zeta} u(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.

**Theorem 4.1.** *Let* D *be a domain in* C *whose boundary consists of* $n \geq 2$ *mutually disjoint Jordan curves and* $\mu : D \to C$ *be measurable function with* $|\mu(z)| < 1$ *a.e. If* $K_\mu$ *satisfies at least one of the conditions from Theorems 3.1–3.3, Corollaries 3.1–3.4, Remarks 3.1 and 3.2, then the Beltrami equation (1.1) has pseudoregular as well as multi-valued solutions of the Dirichlet problem (1.5) for each continuous function* $\phi : \partial D \to R$.

Finally, more refined results on the existence of regular, pseudo-regular and multi-valued solutions of the Dirichlet problem in terms of the so-called tangent dilatations have been proved in the last papers [20] and [21].

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**UDC 511**

## ON THE DISTRIBUTION OF THE EXPONENTIAL DIVISOR FUNCTION

***Lelechenko A. V.***

I.I. Mechnikov Odessa National University, Ukraine

Let $\tau_k^{(e)}$ be a multiplicative function such that $\tau_k^{(e)}\left(p^a\right)=\sum_{d_1\cdots d_k=a} 1$. In the paper the generalizations of $\tau_k^{(e)}$ over the ring of Gaussian integers are introduced. The asymptotic formulas for their average orders are established.

*KEY WORDS*: divisor function, Gaussian integers, asymptotic formula.

## О РАСПРЕДЕЛЕНИИ ЭКСПОНЕНЦИАЛЬНОЙ ФУНКЦИИ ДИВИЗОРОВ

***Лелеченко А.В.***

Пусть $\tau_k^{(e)}$ - мультипликативная функция, такая что $\tau_k^{(e)}\left(p^a\right)=\sum_{d_1\cdots d_k=a} 1$. В работе содержится обобщение $\tau_k^{(e)}$ на кольцо Гауссовых целых чисел. Установлена асимптотическая формула для их средних порядков.

*КЛЮЧЕВЫЕ СЛОВА*: функция делителей, Гауссовы целые числа, асимптотическая формула.

## ПРО РОЗПОДІЛ ЕКСПОНЕНЦІАЛЬНОЇ ФУНКЦІЇ ДИВИЗОРІВ

***Лелеченко А.В.***

Нехай $\tau_k^{(e)}$ – мультиплікативна функція, така що $\tau_k^{(e)}\left(p^a\right)=\sum_{d_1\cdots d_k=a} 1$. В роботі наведено узагальнення $\tau_k^{(e)}$ на кільце Гаусових цілих чисел. Отримана асимптотична формула для їх середніх порядків.

*КЛЮЧОВІ СЛОВА*: функція дільників, Гаусові цілі числа, асимптотична формула.

**1. Introduction**. Exponential divisor function $\tau^{(e)}:\mathbb{Z}\to\mathbb{Z}$ introduced by Subbarao in [7] is a multiplicative function such that

$$\tau^{(e)}(p^a)=\tau(a),$$

where $\tau:\mathbb{Z}\to\mathbb{Z}$ stands for the usual divisor function, $\tau(n)=\sum_{d|n}1$. Erdös estimated its maximal order and Subbarao proved an asymptotic formula for $\sum_{n\le x}\tau^{(e)}(n)$. Later Wu [11] gave more precise estimation:

$$\sum_{n\le x}\tau^{(e)}(n)=Ax+Bx^{1/2}+O\left(x^{\theta_{1,2}+\varepsilon}\right),$$

where $A$ and $B$ are computable constants, $\theta_{1,2}$ is an exponent in the error term of the estimation

$$\sum_{ab^2\le x}1=\zeta(2)x+\zeta(1/2)x^{1/2}+O\left(x^{\theta_{1,2}+\varepsilon}\right).$$

The best modern result is $\theta_{1,2}\le 1057/4785$ [2].

One can consider multidimensional exponential divisor function $\tau_k^{(e)}:\mathbb{Z}\to\mathbb{Z}$ such that

$$\tau_k^{(e)}(p^a)=\tau_k(a),$$

where $\tau_k(n)$ is a number of ordered $k$-tuples of positive integers $(d_1,\ldots,d_k)$ such that $d_1\cdots d_k=n$. So $\tau^{(e)}\equiv\tau_2^{(e)}$. Toth [10] investigated asymptotic properties of $\tau_k^{(e)}$ and proved that for arbitrarily $\varepsilon>0$

$$\sum_{n\le x}\tau_k^{(e)}(n)=C_k x+x^{1/2}S_{k-2}(\log x)+O\left(x^{w_k+\varepsilon}\right),$$

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where $S_{k-2}$ is a polynomial of degree $k-2$ and $w_k = (2k-1)/(4k+1)$.

In the present paper we generalize multidimensional exponential divisor function over the ring of Gaussian integers $\mathbb{Z}[i]$. Namely we introduce multiplicative functions

$\tau_{*k}^{(e)} : \mathbb{Z} \to \mathbb{Z}$, $\mathfrak{t}_k^{(e)} : \mathbb{Z}[i] \to \mathbb{Z}$, $\mathfrak{t}_{*k}^{(e)} : \mathbb{Z}[i] \to \mathbb{Z}$

such that

$$\begin{aligned} \tau_{*k}^{(e)}(p^a) &= \mathfrak{t}_k(a), \\ \mathfrak{t}_k^{(e)}(\mathfrak{p}^a) &= \tau_k(a), \\ \mathfrak{t}_{*k}^{(e)}(\mathfrak{p}^a) &= \mathfrak{t}_k(a), \end{aligned} \tag{1}$$

where $p$ is prime over $\mathbb{Z}$, $\mathfrak{p}$ is prime over $\mathbb{Z}[i]$, $\mathfrak{t}_k(a)$ is a number of ordered $k$-tuples of non-associated in pairs Gaussian integers $(\mathfrak{d}_1, \ldots, \mathfrak{d}_k)$ such that $\mathfrak{d}_1 \cdots \mathfrak{d}_k = a$

The aim of this paper is to provide asymptotic formulas for

$$\sum_{n\le x} \tau_{*k}^{(e)}(n), \quad \sum_{N(\alpha)\le x}{}' \mathfrak{t}_k^{(e)}(\alpha), \quad \sum_{N(\alpha)\le x}{}' \mathfrak{t}_{*k}^{(e)}(\alpha).$$

A theorem on the maximal order of multiplicative functions over $\mathbb{Z}[i]$, generalizing [8], is also proved.

**Notation.** Let us denote the ring of Gaussian integers by $\mathbb{Z}[i]$, $N(a+bi) = a^2 + b^2$.

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols $O$ and $o$, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are written for the argument tending to the infinity.

Letters $\mathfrak{p}$ and $\mathfrak{q}$ with or without indexes denote Gaussian primes; $p$ and $q$ denote rational primes.

As usual $\zeta(s)$ is Riemann zeta-function and $L(s,\chi)$ is Dirichlet *L*-function for some character $\chi$. Let $\chi_4$ be the single nonprincipal character modulo 4, then

$$Z(s) = \zeta(s) L(s, \chi_4)$$

is Hecke zeta-function for the ring of Gaussian integers.

Real and imaginary components of the complex $s$ are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

Notation $\sum'$ means a summation over non-associated elements of $\mathbb{Z}[i]$, and $\prod'$ means the similar relative to multiplication. Notation $a \sim b$ means that $a$ and $b$ are associated, that is $a/b \in \{\pm 1, \pm i\}$. But in asymptotic relations $\sim$ preserve its usual meaning.

Letter $\gamma$ denotes Euler–Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same).

We write $f \star g$ for the notation of the Dirichlet convolution

$$(f \star g)(n) = \sum_{d|n} f(d) g(n/d).$$

**2. Preliminary lemmas.** We need following auxiliary results.

**Lemma 1.** *Gaussian integer* $\mathfrak{p}$ *is prime if and only if one of the following cases complies:*

- $\mathfrak{p} \sim 1+i$,
- $\mathfrak{p} \sim p$, where $p \equiv 3 \pmod 4$,
- $N(\mathfrak{p}) = p$, where $p \equiv 1 \pmod 4$.

*In the last case there are exactly two non-associated* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *such that* $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$.

***Proof.*** See [1].

**Lemma 2.**

$$\sum_{N(\mathfrak{p})\le x}{}' 1 \sim \frac{x}{\log x}, \tag{2}$$

$$\sum_{N(\mathfrak{p})\le x}{}' \log N(\mathfrak{p}) \sim x, \tag{3}$$

***Proof.*** Taking into account Gauss criterion and the asymptotic law of the distribution of primes in the arithmetic progression we have

$$\sum_{N(\mathfrak{p})\le x}{}' 1 \sim \#\{p \mid p \equiv 3 \pmod 4, p \le \sqrt{x}\} +$$
$$+ 2\#\{p \mid p \equiv 1 \pmod 4, p \le x\} \sim$$
$$\sim \frac{\sqrt{x}}{\phi(4)\log x/2} + 2\frac{x}{\phi(4)\log x} = \frac{x}{\log x}.$$

A partial summation gives us the second statement of the lemma.

**Lemma 3.** *Let* $F : \mathbb{Z} \to \mathbb{C}$ *be a multiplicative function such that* $F(p^a) = f(a)$, *where* $f(n) \ll n^\beta$ *for some* $\beta > 0$. *Then*

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_{n\ge 1} \frac{\log f(n)}{n}. \tag{4}$$

***Proof.*** See [8].

**Lemma 4.** *Let* $f(t) \ge 0$. *If*

$$\int_1^T f(t)\,dt \ll g(T),$$

where $g(T) = T^\alpha \log^\beta T$, $\alpha \ge 1$, then

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases} \tag{5}$$

***Proof.*** Let us divide the interval of integration into parts:

$$I(T) \le \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt <$$

$$< \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t)dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates.

**Lemma 5.** *Let* $T > 10$ *and* $|d - 1/2| \ll 1/\log T$. *Then we have the following estimates*

$$\int_{d-iT}^{d+iT} |\zeta(s)|^4 \frac{ds}{s} \ll \log^5 T,$$

$$\int_{d-iT}^{d+iT} |L(s,\chi_4)|^4 \frac{ds}{s} \ll \log^5 T,$$

for growing $T$.

***Proof.*** The statement is the result of the application of Lemma **4** to the estimates [6].

**Lemma 6.** *Let* $\theta > 0$ *be such value that* $\zeta(1/2 + it) \ll t^\theta$ *as* $t \to \infty$, *and let* $\eta > 0$ *be arbitrarily small. Then*

$$\zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\theta)\sigma}, & \sigma \in [0,1/2], \\ |t|^{2\theta(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\theta(1-\sigma)} \log^{2/3}|t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3}|t|, & \sigma \ge 1. \end{cases}$$

The same estimates are valid for $L(s,\chi_4)$ also.

***Proof.*** The statement follows from Phragmén—Lindelöf principle, exact and approximate functional equations for $\zeta(s)$ and $L(s,\chi_4)$. See [4] and [9] for details.

The best modern result [3] is that $\theta \le 32/205 + \varepsilon$. If Riemann hypothesis holds for $\zeta$ and for $L(s,\chi_4)$ then $\theta \le \varepsilon$.

**3. Main results.** The following theorem generalizes Lemma **3** to Gaussian integers; the proof's outline follows the proof of Lemma **3** in [8].

**Theorem 7.** *Let* $F : \mathbb{Z}[i] \to \mathbb{C}$ *be a multiplicative function such that* $F(\mathfrak{p}^a) = f(a)$, *where* $f(n) \ll n^\beta$ *for some* $\beta > 0$. *Then*

$$\limsup_{\alpha\to\infty} \frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} = \sup_{n\ge1} \frac{\log f(n)}{n} := K_f. \quad (6)$$

***Proof.*** Let us fix arbitrarily small $\varepsilon > 0$.

Firstly, let us show that there are infinitely many $\alpha$ such that

$$\frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} > K_f - \varepsilon.$$

By definition of $K_f$ we can choose $l$ such that

$$(\log f(l))/l > K_f - \varepsilon/2.$$

It follows from (3) that for $x \ge 2$ inequality

$$\sum\nolimits'_{N(\mathfrak{p})\le x} \log N(\mathfrak{p}) > Ax$$

holds, where $0 < A < 1$.

Let $\mathfrak{q}$ be an arbitrarily large Gaussian prime, $N(\mathfrak{q}) \ge 2$. Consider

$$r = \sum\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} 1, \qquad \alpha = \prod\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} \mathfrak{p}^l.$$

Then $F_k(\alpha) = (f(l))^r$ and we have

$$r \log N(\mathfrak{q}) \ge \frac{\log N(\alpha)}{l} = \sum\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} \log N(\mathfrak{p}) > AN(\mathfrak{q}), \quad (7)$$

$$\log F(\alpha) = r \log f(l) \ge \frac{\log N(\alpha)}{\log N(\mathfrak{q})} \frac{\log f(l)}{l}. \quad (8)$$

But (7) implies

$$\log A + \log N(\mathfrak{q}) < \log \frac{\log N(\alpha)}{l} \le \log\log N(\alpha),$$

so $\log N(\mathfrak{q}) < \log\log N(\alpha) - \log A$. Then it follows from (8) that

$$\log F(\alpha) > \frac{\log N(\alpha)}{\log\log N(\alpha) - \log A} \frac{\log f(l)}{l}$$

and since $(\log f(l))/l > K_f - \varepsilon/2$ and $A < 1$ we have

$$\frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} > \frac{\log\log N(\alpha)}{\log\log N(\alpha) - \log A} \times$$
$$\times (K_f - \varepsilon/2) > K_f - \varepsilon.$$

Secondly, let us show the existence of $N(\varepsilon)$ such that for all $n \ge N(\varepsilon)$ we have

$$\frac{\log F(n) \log\log N(\alpha)}{\log N(\alpha)} < (1+\varepsilon)K_f.$$

Let us choose $\delta \in (0,\varepsilon)$ and $\eta \in (0, \delta/(1+\delta))$. Suppose $N(\alpha) \ge 3$, then we define

$$\omega := \omega(\alpha) = \frac{(1+\delta)K_f}{\log\log N(\alpha)}, \quad \Omega := \Omega(\alpha) = \log^{1-\eta} N(\alpha).$$

By choice of $\delta$ and $\eta$ we have

$$\Omega^\omega = \exp(\omega \log \Omega) = \exp\left((1-\eta)(1+\delta)K_f\right) > e^{K_f}.$$

Suppose that the canonical expansion of $\alpha$ is

$$\alpha \sim \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r} \mathfrak{q}_1^{b_1} \cdots \mathfrak{q}_s^{b_s},$$

where $N(\mathfrak{p}_k) \le \Omega$ and $N(\mathfrak{q}_k) > \Omega$. Then

$$\frac{F(\alpha)}{N^\omega(\alpha)} = \prod_{k=1}^r \frac{f(a_k)}{N^{\omega a_k}(\mathfrak{p}_k)} \cdot \prod_{k=1}^s \frac{f(b_k)}{N^{\omega b_k}(\mathfrak{q}_k)} := \Pi_1 \cdot \Pi_2. \quad (9)$$

Since $\Omega^\omega > e^{K_f}$ and $K_f \ge (\log f(b_k))/b_k$ then

$$\frac{f(b_k)}{N^{\omega b_k}(\mathfrak{q}_k)} < \frac{f(b_k)}{\Omega^{\omega b_k}} < \frac{f(b_k)}{e^{K_f b_k}} \le 1$$

and it follows that $\Pi_2 \le 1$. Consider $\Pi_1$. From the statement of the theorem we have $f(n) \ll n^\beta$, so

$$\frac{f(a_k)}{N^{\omega a_k}(\mathfrak{p}_k)} \ll \frac{a_k^\beta}{(a_k\omega)^\beta} \ll \omega^{-\beta}.$$

Then

$$\log \Pi_1 \ll \Omega \log w^{-\beta} \ll$$
$$\ll \log^{1-\eta} N(\alpha) \log\log\log N(\alpha) =$$
$$= o\left(\frac{\log N(\alpha)}{\log\log N(\alpha)}\right).$$

Finally by (9) we get

$$\log F(n) = \omega \log n + \log \Pi_1 + \log \Pi_2 =$$
$$= \frac{(1+\delta)K_f \log n}{\log\log n} + \frac{(\varepsilon-\delta)K_f \log n}{\log\log n}.$$

**Lemma 8.**

$$\tau_{*k}^{(e)}(n) \ll n^{\varepsilon},$$
$$t_k^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha), \quad (10)$$
$$t_{*k}^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha).$$

***Proof.*** Taking into account trivial estimates $\tau_k(n) \le n$ and $t_k(n) \le n^2$ we have that

$$\sup_{n\ge1} \log \tau_k(n) n < \infty, \qquad \sup_{n\ge1} \log t_k(n) n < \infty.$$

Now the estimates (10) follows from Theorem **7** and Lemma **3**.

We are ready to provide asymptotic formulas for sums of $\tau_{*k}^{(e)}(n)$, $t_k^{(e)}(\alpha)$, $t_{*k}^{(e)}(\alpha)$. Let us denote

$$G_{*k}(s) := \sum_n \tau_{*k}^{(e)}(n) n^{-s}, \qquad T_{*k}(x) := \sum_{n\le x} \tau_{*k}^{(e)}(n),$$
$$F_k(s) := \sum_\alpha{}' t_k^{(e)}(\alpha) N^{-s}(\alpha), \quad M_k(x) := \sum_{N(\alpha)\le x}{}' t_k^{(e)}(\alpha),$$
$$F_{*k}(s) := \sum_\alpha{}' t_{*k}^{(e)}(\alpha) N^{-s}(\alpha), \; M_{*k}(x) := \sum_{N(\alpha)\le x}{}' t_{*k}^{(e)}(\alpha).$$

**Lemma 9.**

$$G_{*k}(s) \;=\; \zeta(s)\zeta^{(k^2+k-2)/2}(2s)\zeta^{(-k^2+k)/2}(3s)\times$$
$$\times\zeta^{(-k^4+7k^2-6k)/12}(4s)\times \quad (11)$$
$$\times\zeta^{(5k^4-6k^3-5k^2+6k)/24}(5s)K_{*k}(s),$$

$$F_k(s) \;=\; Z(s)Z^{k-1}(2s)Z^{(k-k^2)/2}(5s)\times$$
$$\times Z^{(-k^3+6k^2-5k)/6}(6s)\times Z^{(k^3-4k^2+3k)/2}(7s)\times \quad (12)$$
$$\times Z^{(3k^4-26k^3+57k^2-34k)/24}(8s)H_k(s),$$

$$F_{*k}(s) = Z(s)Z^{(k^2+k-2)/2}(2s)Z^{(-k^2+k)/2}(3s)\times$$
$$\times Z^{(-k^4+7k^2-6k)/12}(4s)\times \quad (13)$$
$$\times Z^{(5k^4-6k^3-5k^2+6k)/24}(5s)H_{*k}(s),$$

where Dirichlet series $H(s)$ are absolutely convergent for $\Re s > 1/9$ and Dirichlet series for $H_*(s)$, $K_*(s)$ are absolutely convergent for $\Re s > 1/6$.

***Proof.*** The statements can be verified by direct computation of Bell series of corresponding functions. For example, Bell series for $t_k^{(e)}$ have the following representation:

$$\left(\sum_{a=0}^{\infty} t_k^{(e)}(\mathfrak{p}^a)x^a\right)(1-x)(1-x^2)^{k-1}(1-x^5)^{(k-k^2)/2}\times$$
$$\times(1-x^6)^{(-k^3+6k^2-5k)/6}\times(1-x^7)^{(k^3-4k^2+3k)/2}\times$$
$$\times(1-x^8)^{(3k^4-26k^3+57k^2-34k)/24} = 1+O(x^9).$$

**Theorem 10.**

$$T_{*k}(x) = A_k x + x^{1/2}P_k(\log x)O(x^{w_k+\varepsilon}), \quad (14)$$

where $P_k$ is a polynomial, $\deg P_k = (k^2+k-4)/2$, and

$$w_k = \frac{k^2+k-1}{2k^2+2k+1}.$$

***Proof.*** Let $l = (k^2+k-2)/2$, $\mathbf{a} = (1, \underbrace{2,\ldots,2}_{l})$.

Identity (11) implies

$$\tau_{*k}^{(e)} = \tau(\mathbf{a};\cdot)\star f, \qquad T_{*k}(x) = \sum_{n\le x} T(\mathbf{a}; x/n)f(n) \quad (15)$$

where

$$\tau(\mathbf{a}; n) = \sum_{d_0 d_1^2\cdots d_l^2 = n} 1,$$
$$T(\mathbf{a}; x) := \sum_{n\le x}\tau(\mathbf{a}; n) = \sum_{d_0 d_1^2\cdots d_l \le x} 1,$$

and the series $\sum_{n=1}^{\infty} f(n)n^{-\sigma}$ are absolutely convergent for $\sigma > 1/3$. Due to [5] we have

$$T(\mathbf{a}; x) = C_1 x + x^{1/2}Q(\log x) + O(x^{w_k+\varepsilon}), \quad (16)$$

where $Q$ is a polynomial, $\deg Q = l-1$, and

$$w_k = \frac{2l+1}{4l+5}.$$

For $k \ge 2$ we have $w_k > 1/3$.

One can get the following estimates:

$$\sum_{n>x}\frac{f(n)}{n} = O\left(x^{-2/3+\varepsilon}\sum_{n>x}\frac{f(n)}{n^{1/3+\varepsilon}}\right) = O(x^{-2/3+\varepsilon}), \quad (17)$$

$$\sum_{n>x}\frac{f(n)\log^a n}{n^{1/2}} = O\left(x^{-1/6+\varepsilon}\sum_{n>x}\frac{f(n)\log^a n}{n^{1/3+\varepsilon}}\right) = O(x^{-1/6+\varepsilon}). \quad (18)$$

for $a \ge 0$.

Finally, substituting estimates (16), (17) and (18) into (15) we get

$$T_{*k}(x) = C_1 x\sum_{n\le x}\frac{f(n)}{n} + x^{1/2}\sum_{n\le x}\frac{f(n)Q(\log(x/n))}{n^{1/2}} +$$
$$O(x^{w_k+\varepsilon}) = A_k x + x^{1/2}P_k(\log x) + O(x^{w_k+\varepsilon}).$$

**Lemma 11.**

$$\operatorname*{res}_{s=1} F_k(s)x^s/s = C_k x, \qquad \operatorname*{res}_{s=1} F_{*k}(s)x^s/s = C_{*k}x, \quad (19)$$

where

$$C_k = \frac{\pi}{4}\prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^a(\mathfrak{p})}\right), \quad (20)$$

$$C_{*k} = \frac{\pi}{4}\prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{t_k(a)-t_k(a-1)}{N^a(\mathfrak{p})}\right). \quad (21)$$

***Proof.*** As a consequence of the representation (12) we have

$$\frac{F_k(s)}{Z(s)} = \prod_{\mathfrak{p}}\left(1+\sum_{a=1}^{\infty}\frac{\tau_k(a)}{N^{as}(\mathfrak{p})}\right)(1-\mathfrak{p}^{-1}) =$$

$$= \prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^{as}(\mathfrak{p})}\right),$$

and so function $F_k(s)/Z(s)$ is regular in the neighbourhood of $s=1$. At the same time we have

$$\operatorname*{res}_{s=1} Z(s) = L(1,\chi_4)\operatorname*{res}_{s=1}\zeta(s) = \frac{\pi}{4},$$

which implies (20). The proof of (21) is similar.

**Theorem 12.**

$$M_k(x) = C_k x + O(x^{1/2}\log^{3+4(k-1)/3}x), \quad (22)$$

$$M_{*k}(x) = C_{*k} x + O(x^{1/2}\log^{3+2(k^2+k-2)/3}x), \quad (23)$$

where $C_k$ and $C_{*k}$ were defined in (20) and (21).

***Proof.*** By Perron formula and by (10) for $c = 1+1/\log x$, $\log T \asymp \log x$ we have

$$M_k(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_k(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

Suppose $d = 1/2 - 1/\log x$. Let us shift the interval of integration to $[d-iT, d+iT]$. To do this consider an integral about a closed rectangle path with vertexes in $d-iT$, $d+iT$, $c+iT$ and $c-iT$. There are two poles in $s=1$ and $s=1/2$ inside the contour. The residue at $s=1$ was calculated in (19). The residue at $s=1/2$ is equal to $Dx^{1/2}$, *D* is constant, and will be absorbed by error term (see below).

Identity (12) implies

$$F_k(s) = Z(s)Z^{k-1}(2s)L_k(s),$$

where $L_k(s)$ is regular for $\Re s > 1/3$, so for each $\varepsilon > 0$ it is uniformly bounded for $\Re s > 1/3+\varepsilon$.

Let us estimate the error term using Lemma **5** and Lemma **6**. The error term absorbs values of integrals about three sides of the integration's rectangle. We take into account $Z(s) = \zeta(s)L(s,\chi_4)$. On the horizontal segments we have

$$\int_{d+iT}^{c+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll$$

$$\ll \max_{\sigma\in[d,c]} Z(\sigma+iT)Z^{k-1}(2\sigma+2iT)x^{\sigma}T^{-1} \ll$$

$$\ll x^{1/2}T^{2\theta-1}\log^{4(k-1)/3}T + xT^{-1}\log^{4/3}T,$$

It is well-known that $\zeta(s) \sim (s-1)^{-1}$ in the neighborhood of $s=1$. So on the vertical segment we have the following estimates. Near pole one can calculate that

$$\int_d^{d+i} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll x^{1/2}\int_0^1 \zeta^{k-1}(2d+2it)dt \ll$$

$$\ll x^{1/2}\int_0^1 \frac{dt}{|it-1/\log x|^{k-1}} \ll x^{1/2}\log^{k-1}x,$$

and for the rest of the vertical segment we get

$$\int_{d+i}^{d+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll$$

$$\ll \left(\int_1^T |\zeta(1/2+it)|^4\frac{dt}{t}\int_1^T |L(1/2+it,\chi_4)|^4\frac{dt}{t}\right)^{1/4} \times$$

$$\times\left(\int_1^T |Z(1+2it)|^{2(k-1)}\frac{dt}{t}\right)^{1/2} \ll$$

$$\ll x^{1/2}\left(\log^5 T\cdot\log^{8(k-1)/3+1}T\right)^{1/2} \ll$$

$$\ll x^{1/2}\log^{3+4(k-1)/3}T.$$

The choice $T = x^{1/2+\varepsilon}$ finishes the proof of (22).

The proof of (23) is similar, but due to (13) one have replace $k-1$ by $(k^2+k-2)/2$.


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