UDC 511

ON THE DISTRIBUTION OF THE EXPONENTIAL DIVISOR FUNCTION

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Let $\tau_k^{(e)}$ be a multiplicative function such that $\tau_k^{(e)}\left(p^a\right) = \sum_{d_1 \cdots d_k = a} 1$. In the paper the generalizations of $\tau_k^{(e)}$ over the ring of Gaussian integers are introduced. The asymptotic formulas for their average orders are established.

KEY WORDS: divisor function, Gaussian integers, asymptotic formula.

О РАСПРЕДЕЛЕНИИ ЭКСПОНЕНЦИАЛЬНОЙ ФУНКЦИИ ДИВИЗОРОВ

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Пусть $\tau_k^{(e)}$ - мультипликативная функция, такая что $\tau_k^{(e)}\left(p^a\right) = \sum_{d_1 \cdots d_k = a} 1$. В работе содержится обобщение $\tau_k^{(e)}$ на кольцо Гауссовых целых чисел. Установлена асимптотическая формула для их средних порядков.

КЛЮЧЕВЫЕ СЛОВА: функция делителей, Гауссовы целые числа, асимптотическая формула.

ПРО РОЗПОДІЛ ЕКСПОНЕНЦІАЛЬНОЇ ФУНКЦІЇ ДИВИЗОРІВ

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Нехай $\tau_k^{(e)}$ – мультиплікативна функція, така що $\tau_k^{(e)}\left(p^a\right) = \sum_{d_1 \cdots d_k = a} 1$. В роботі наведено узагальнення $\tau_k^{(e)}$ на кільце Гаусових цілих чисел. Отримана асимптотична формула для їх середніх порядків.

КЛЮЧОВІ СЛОВА: функція дільників, Гаусові цілі числа, асимптотична формула.

1. Introduction. Exponential divisor function $\tau^{(e)} : \mathbb{Z} \to \mathbb{Z}$ introduced by Subbarao in [7] is a multiplicative function such that

$$\tau^{(e)}(p^a) = \tau(a),$$

where $\tau : \mathbb{Z} \to \mathbb{Z}$ stands for the usual divisor function, $\tau(n) = \sum_{d|n} 1$. Erdös estimated its maximal order and Subbarao proved an asymptotic formula for $\sum_{n \le x} \tau^{(e)}(n)$. Later Wu [11] gave more precise estimation:

$$\sum_{n \le x} \tau^{(e)}(n) = Ax + Bx^{1/2} + O\left(x^{\theta_{1,2}+\varepsilon}\right),$$

where A and B are computable constants, $\theta_{1,2}$ is an exponent in the error term of the estimation

$$\sum_{ab^2 \le x} 1 = \zeta(2)x + \zeta(1/2)x^{1/2} + O\left(x^{\theta_{1,2}+\varepsilon}\right).$$

The best modern result is $\theta_{1,2} \le 1057/4785$ [2].

One can consider multidimensional exponential divisor function $\tau_k^{(e)} : \mathbb{Z} \to \mathbb{Z}$ such that

$$\tau_k^{(e)}(p^a) = \tau_k(a),$$

where $\tau_k(n)$ is a number of ordered k-tuples of positive integers $(d_1, \ldots, d_k)$ such that $d_1 \cdots d_k = n$. So $\tau^{(e)} \equiv \tau_2^{(e)}$. Toth [10] investigated asymptotic properties of $\tau_k^{(e)}$ and proved that for arbitrarily $\varepsilon > 0$

$$\sum_{n \le x} \tau_k^{(e)}(n) = C_k x + x^{1/2} S_{k-2}(\log x) + O\left(x^{w_k+\varepsilon}\right),$$

where S_{k-2} is a polynomial of degree $k-2$ and $w_k = (2k-1)/(4k+1)$.

In the present paper we generalize multidimensional exponential divisor function over the ring of Gaussian integers $\mathbb{Z}[i]$. Namely we introduce multiplicative functions

$\tau_{*k}^{(e)} : \mathbb{Z} \to \mathbb{Z}$, $\quad \mathfrak{t}_k^{(e)} : \mathbb{Z}[i] \to \mathbb{Z}$, $\quad \mathfrak{t}_{*k}^{(e)} : \mathbb{Z}[i] \to \mathbb{Z}$

such that

$$\begin{aligned} \tau_{*k}^{(e)}(p^a) &= \mathfrak{t}_k(a), \\ \mathfrak{t}_k^{(e)}(\mathfrak{p}^a) &= \tau_k(a), \\ \mathfrak{t}_{*k}^{(e)}(\mathfrak{p}^a) &= \mathfrak{t}_k(a), \end{aligned} \tag{1}$$

where p is prime over $\mathbb{Z}$, $\mathfrak{p}$ is prime over $\mathbb{Z}[i]$, $\mathfrak{t}_k(a)$ is a number of ordered k-tuples of non-associated in pairs Gaussian integers $(\mathfrak{d}_1, \ldots, \mathfrak{d}_k)$ such that $\mathfrak{d}_1 \cdots \mathfrak{d}_k = a$

The aim of this paper is to provide asymptotic formulas for

$$\sum_{n \le x} \tau_{*k}^{(e)}(n), \quad \sideset{}{'}\sum_{N(\alpha) \le x} \mathfrak{t}_k^{(e)}(\alpha), \quad \sideset{}{'}\sum_{N(\alpha) \le x} \mathfrak{t}_{*k}^{(e)}(\alpha).$$

A theorem on the maximal order of multiplicative functions over $\mathbb{Z}[i]$, generalizing [8], is also proved.

Notation. Let us denote the ring of Gaussian integers by $\mathbb{Z}[i]$, $N(a+bi) = a^2 + b^2$.

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are written for the argument tending to the infinity.

Letters $\mathfrak{p}$ and $\mathfrak{q}$ with or without indexes denote Gaussian primes; p and q denote rational primes.

As usual $\zeta(s)$ is Riemann zeta-function and $L(s, \chi)$ is Dirichlet *L*-function for some character χ. Let χ_4 be the single nonprincipal character modulo 4, then

$$Z(s) = \zeta(s) L(s, \chi_4)$$

is Hecke zeta-function for the ring of Gaussian integers.

Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

Notation $\sum'$ means a summation over non-associated elements of $\mathbb{Z}[i]$, and $\prod'$ means the similar relative to multiplication. Notation $a \sim b$ means that a and b are associated, that is $a/b \in \{\pm 1, \pm i\}$. But in asymptotic relations $\sim$ preserve its usual meaning.

Letter γ denotes Euler–Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same).

We write $f \star g$ for the notation of the Dirichlet convolution

$$(f \star g)(n) = \sum_{d \mid n} f(d) g(n/d).$$

2. Preliminary lemmas. We need following auxiliary results.

Lemma 1. *Gaussian integer $\mathfrak{p}$ is prime if and only if one of the following cases complies:*

- $\mathfrak{p} \sim 1 + i$,
- $\mathfrak{p} \sim p$, where $p \equiv 3 \pmod 4$,
- $N(\mathfrak{p}) = p$, where $p \equiv 1 \pmod 4$.

In the last case there are exactly two non-associated $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$.

Proof. See [1].

Lemma 2.

$$\sideset{}{'}\sum_{N(\mathfrak{p}) \le x} 1 \sim \frac{x}{\log x}, \tag{2}$$

$$\sideset{}{'}\sum_{N(\mathfrak{p}) \le x} \log N(\mathfrak{p}) \sim x, \tag{3}$$

Proof. Taking into account Gauss criterion and the asymptotic law of the distribution of primes in the arithmetic progression we have

$$\sideset{}{'}\sum_{N(\mathfrak{p}) \le x} 1 \sim \#\{p \mid p \equiv 3 \pmod 4, p \le \sqrt{x}\} + $$
$$+ 2\#\{p \mid p \equiv 1 \pmod 4, p \le x\} \sim$$
$$\sim \frac{\sqrt{x}}{\phi(4) \log x / 2} + 2\frac{x}{\phi(4) \log x} = \frac{x}{\log x}.$$

A partial summation gives us the second statement of the lemma.

Lemma 3. *Let $F : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function such that $F(p^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then*

$$\limsup_{n \to \infty} \frac{\log F(n) \log \log n}{\log n} = \sup_{n \ge 1} \frac{\log f(n)}{n}. \tag{4}$$

Proof. See [8].

Lemma 4. *Let $f(t) \ge 0$. If*

$$\int_1^T f(t)\,dt \ll g(T),$$

where $g(T) = T^\alpha \log^\beta T$, $\alpha \ge 1$, then

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases} \tag{5}$$

Proof. Let us divide the interval of integration into parts:

$$I(T) \le \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt <$$

$$< \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates.

Lemma 5. *Let* $T > 10$ *and* $|d - 1/2| \ll 1/\log T$. *Then we have the following estimates*

$$\int_{d-iT}^{d+iT} |\zeta(s)|^4 \frac{ds}{s} \ll \log^5 T,$$

$$\int_{d-iT}^{d+iT} |L(s,\chi_4)|^4 \frac{ds}{s} \ll \log^5 T,$$

for growing T.

Proof. The statement is the result of the application of Lemma **4** to the estimates [6].

Lemma 6. *Let* $\theta > 0$ *be such value that* $\zeta(1/2 + it) \ll t^\theta$ *as* $t \to \infty$, *and let* $\eta > 0$ *be arbitrarily small. Then*

$$\zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\theta)\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\theta(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\theta(1-\sigma)} \log^{2/3}|t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3}|t|, & \sigma \ge 1. \end{cases}$$

The same estimates are valid for $L(s,\chi_4)$ also.

Proof. The statement follows from Phragmén—Lindelöf principle, exact and approximate functional equations for $\zeta(s)$ and $L(s,\chi_4)$. See [4] and [9] for details.

The best modern result [3] is that $\theta \le 32/205 + \varepsilon$. If Riemann hypothesis holds for ζ and for $L(s,\chi_4)$ then $\theta \le \varepsilon$.

3. Main results. The following theorem generalizes Lemma **3** to Gaussian integers; the proof's outline follows the proof of Lemma **3** in [8].

Theorem 7. *Let* $F : \mathbb{Z}[i] \to \mathbb{C}$ *be a multiplicative function such that* $F(\mathfrak{p}^a) = f(a)$, *where* $f(n) \ll n^\beta$ *for some* $\beta > 0$. *Then*

$$\limsup_{\alpha\to\infty} \frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} = \sup_{n\ge1} \frac{\log f(n)}{n} := K_f. \quad (6)$$

Proof. Let us fix arbitrarily small $\varepsilon > 0$.

Firstly, let us show that there are infinitely many α such that

$$\frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} > K_f - \varepsilon.$$

By definition of K_f we can choose l such that

$$(\log f(l))/l > K_f - \varepsilon/2.$$

It follows from (3) that for $x \ge 2$ inequality

$$\sum\nolimits'_{N(\mathfrak{p})\le x} \log N(\mathfrak{p}) > Ax$$

holds, where $0 < A < 1$.

Let $\mathfrak{q}$ be an arbitrarily large Gaussian prime, $N(\mathfrak{q}) \ge 2$. Consider

$$r = \sum\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} 1, \qquad \alpha = \prod\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} \mathfrak{p}^l.$$

Then $F_k(\alpha) = (f(l))^r$ and we have

$$r \log N(\mathfrak{q}) \ge \frac{\log N(\alpha)}{l} = \sum\nolimits'_{N(\mathfrak{p})\le N(\mathfrak{q})} \log N(\mathfrak{p}) > AN(\mathfrak{q}), \quad (7)$$

$$\log F(\alpha) = r \log f(l) \ge \frac{\log N(\alpha)}{\log N(\mathfrak{q})} \frac{\log f(l)}{l}. \quad (8)$$

But (7) implies

$$\log A + \log N(\mathfrak{q}) < \log \frac{\log N(\alpha)}{l} \le \log\log N(\alpha),$$

so $\log N(\mathfrak{q}) < \log\log N(\alpha) - \log A$. Then it follows from (8) that

$$\log F(\alpha) > \frac{\log N(\alpha)}{\log\log N(\alpha) - \log A} \frac{\log f(l)}{l}$$

and since $(\log f(l))/l > K_f - \varepsilon/2$ and $A < 1$ we have

$$\frac{\log F(\alpha) \log\log N(\alpha)}{\log N(\alpha)} > \frac{\log\log N(\alpha)}{\log\log N(\alpha) - \log A} \times$$
$$\times (K_f - \varepsilon/2) > K_f - \varepsilon.$$

Secondly, let us show the existence of $N(\varepsilon)$ such that for all $n \ge N(\varepsilon)$ we have

$$\frac{\log F(n) \log\log N(\alpha)}{\log N(\alpha)} < (1+\varepsilon) K_f.$$

Let us choose $\delta \in (0, \varepsilon)$ and $\eta \in (0, \delta/(1+\delta))$. Suppose $N(\alpha) \ge 3$, then we define

$$\omega := \omega(\alpha) = \frac{(1+\delta)K_f}{\log\log N(\alpha)}, \quad \Omega := \Omega(\alpha) = \log^{1-\eta} N(\alpha).$$

By choice of δ and η we have

$$\Omega^\omega = \exp(\omega \log \Omega) = \exp\big((1-\eta)(1+\delta)K_f\big) > e^{K_f}.$$

Suppose that the canonical expansion of α is

$$\alpha \sim \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r} \mathfrak{q}_1^{b_1} \cdots \mathfrak{q}_s^{b_s},$$

where $N(\mathfrak{p}_k) \le \Omega$ and $N(\mathfrak{q}_k) > \Omega$. Then

$$\frac{F(\alpha)}{N^\omega(\alpha)} = \prod_{k=1}^r \frac{f(a_k)}{N^{\omega a_k}(\mathfrak{p}_k)} \cdot \prod_{k=1}^s \frac{f(b_k)}{N^{\omega b_k}(\mathfrak{q}_k)} := \Pi_1 \cdot \Pi_2. \quad (9)$$

Since $\Omega^\omega > e^{K_f}$ and $K_f \ge (\log f(b_k))/b_k$ then

$$\frac{f(b_k)}{N^{\omega b_k}(\mathfrak{q}_k)} < \frac{f(b_k)}{\Omega^{\omega b_k}} < \frac{f(b_k)}{e^{K_f b_k}} \le 1$$

and it follows that $\Pi_2 \le 1$. Consider Π_1. From the statement of the theorem we have $f(n) \ll n^\beta$, so

$$\frac{f(a_k)}{N^{\omega a_k}(\mathfrak{p}_k)} \ll \frac{a_k^\beta}{(a_k \omega)^\beta} \ll \omega^{-\beta}.$$

Then

$$\log \Pi_1 \ll \Omega \log w^{-\beta} \ll$$
$$\ll \log^{1-\eta} N(\alpha) \log\log\log N(\alpha) =$$
$$= o\left(\frac{\log N(\alpha)}{\log\log N(\alpha)}\right).$$

Finally by (9) we get

$$\log F(n) = \omega \log n + \log \Pi_1 + \log \Pi_2 =$$
$$= \frac{(1+\delta)K_f \log n}{\log\log n} + \frac{(\varepsilon-\delta)K_f \log n}{\log\log n}.$$

Lemma 8.

$$\tau_{*k}^{(e)}(n) \ll n^{\varepsilon},$$
$$t_k^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha), \qquad (10)$$
$$t_{*k}^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha).$$

Proof. Taking into account trivial estimates $\tau_k(n) \le n$ and $t_k(n) \le n^2$ we have that

$$\sup_{n\ge 1} \log \tau_k(n) n < \infty, \qquad \sup_{n\ge 1} \log t_k(n) n < \infty.$$

Now the estimates (10) follows from Theorem **7** and Lemma **3**.

We are ready to provide asymptotic formulas for sums of $\tau_{*k}^{(e)}(n)$, $t_k^{(e)}(\alpha)$, $t_{*k}^{(e)}(\alpha)$. Let us denote

$$G_{*k}(s) := \sum_n \tau_{*k}^{(e)}(n) n^{-s}, \qquad T_{*k}(x) := \sum_{n\le x} \tau_{*k}^{(e)}(n),$$
$$F_k(s) := \sum_{\alpha} t_k^{(e)}(\alpha) N^{-s}(\alpha), \quad M_k(x) := \sum_{N(\alpha)\le x} t_k^{(e)}(\alpha),$$
$$F_{*k}(s) := \sum_{\alpha} t_{*k}^{(e)}(\alpha) N^{-s}(\alpha), \quad M_{*k}(x) := \sum_{N(\alpha)\le x} t_{*k}^{(e)}(\alpha).$$

Lemma 9.

$$G_{*k}(s) = \zeta(s)\zeta^{(k^2+k-2)/2}(2s)\zeta^{(-k^2+k)/2}(3s)\times$$
$$\times\zeta^{(-k^4+7k^2-6k)/12}(4s)\times \qquad (11)$$
$$\times\zeta^{(5k^4-6k^3-5k^2+6k)/24}(5s)K_{*k}(s),$$

$$F_k(s) = Z(s)Z^{k-1}(2s)Z^{(k-k^2)/2}(5s)\times$$
$$\times Z^{(-k^3+6k^2-5k)/6}(6s)\times Z^{(k^3-4k^2+3k)/2}(7s)\times \qquad (12)$$
$$\times Z^{(3k^4-26k^3+57k^2-34k)/24}(8s)H_k(s),$$

$$F_{*k}(s) = Z(s)Z^{(k^2+k-2)/2}(2s)Z^{(-k^2+k)/2}(3s)\times$$
$$\times Z^{(-k^4+7k^2-6k)/12}(4s)\times \qquad (13)$$
$$\times Z^{(5k^4-6k^3-5k^2+6k)/24}(5s)H_{*k}(s),$$

where Dirichlet series $H(s)$ are absolutely convergent for $\Re s > 1/9$ and Dirichlet series for $H_*(s)$, $K_*(s)$ are absolutely convergent for $\Re s > 1/6$.

Proof. The statements can be verified by direct computation of Bell series of corresponding functions. For example, Bell series for $t_k^{(e)}$ have the following representation:

$$\left(\sum_{a=0}^{\infty} t_k^{(e)}(p^a)x^a\right)(1-x)(1-x^2)^{k-1}(1-x^5)^{(k-k^2)/2}\times$$
$$\times(1-x^6)^{(-k^3+6k^2-5k)/6}\times(1-x^7)^{(k^3-4k^2+3k)/2}\times$$
$$\times(1-x^8)^{(3k^4-26k^3+57k^2-34k)/24} = 1+O(x^9).$$

Theorem 10.

$$T_{*k}(x) = A_k x + x^{1/2}P_k(\log x)O(x^{w_k+\varepsilon}), \quad (14)$$

where P_k is a polynomial, $\deg P_k = (k^2+k-4)/2$, and

$$w_k = \frac{k^2+k-1}{2k^2+2k+1}.$$

Proof. Let $l = (k^2+k-2)/2$, $\mathbf{a} = (1, \underbrace{2,\ldots,2}_{l})$.

Identity (11) implies

$$\tau_{*k}^{(e)} = \tau(\mathbf{a};\cdot)\star f, \qquad T_{*k}(x) = \sum_{n\le x} T(\mathbf{a}; x/n) f(n) \quad (15)$$

where

$$\tau(\mathbf{a}; n) = \sum_{d_0 d_1^2 \cdots d_l^2 = n} 1,$$
$$T(\mathbf{a}; x) := \sum_{n\le x} \tau(\mathbf{a}; n) = \sum_{d_0 d_1^2 \cdots d_l \le x} 1,$$

and the series $\sum_{n=1}^{\infty} f(n) n^{-\sigma}$ are absolutely convergent for $\sigma > 1/3$. Due to [5] we have

$$T(\mathbf{a}; x) = C_1 x + x^{1/2} Q(\log x) + O(x^{w_k+\varepsilon}), \quad (16)$$

where Q is a polynomial, $\deg Q = l-1$, and

$$w_k = \frac{2l+1}{4l+5}.$$

For $k \ge 2$ we have $w_k > 1/3$.

One can get the following estimates:

$$\sum_{n>x} \frac{f(n)}{n} = O\left(x^{-2/3+\varepsilon} \sum_{n>x} \frac{f(n)}{n^{1/3+\varepsilon}}\right) = O(x^{-2/3+\varepsilon}), \quad (17)$$

$$\sum_{n>x} \frac{f(n)\log^a n}{n^{1/2}} = O\left(x^{-1/6+\varepsilon} \sum_{n>x} \frac{f(n)\log^a n}{n^{1/3+\varepsilon}}\right) = O(x^{-1/6+\varepsilon}). \quad (18)$$

for $a \ge 0$.

Finally, substituting estimates (16), (17) and (18) into (15) we get

$$T_{*k}(x) = C_1 x \sum_{n\le x} \frac{f(n)}{n} + x^{1/2} \sum_{n\le x} \frac{f(n)Q(\log(x/n))}{n^{1/2}} +$$
$$O(x^{w_k+\varepsilon}) = A_k x + x^{1/2} P_k(\log x) + O(x^{w_k+\varepsilon}).$$

Lemma 11.

$$\operatorname*{res}_{s=1} F_k(s) x^s / s = C_k x, \qquad \operatorname*{res}_{s=1} F_{*k}(s) x^s / s = C_{*k} x, \quad (19)$$

where

$$C_k = \frac{\pi}{4}\prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^a(\mathfrak{p})}\right), \quad (20)$$

$$C_{*k} = \frac{\pi}{4}\prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{t_k(a)-t_k(a-1)}{N^a(\mathfrak{p})}\right). \quad (21)$$

Proof. As a consequence of the representation (12) we have

$$\frac{F_k(s)}{Z(s)} = \prod_{\mathfrak{p}}\left(1+\sum_{a=1}^{\infty}\frac{\tau_k(a)}{N^{as}(\mathfrak{p})}\right)(1-\mathfrak{p}^{-1}) =$$

$$= \prod_{\mathfrak{p}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^{as}(\mathfrak{p})}\right),$$

and so function $F_k(s)/Z(s)$ is regular in the neighbourhood of $s=1$. At the same time we have

$$\operatorname*{res}_{s=1} Z(s) = L(1,\chi_4)\operatorname*{res}_{s=1}\zeta(s) = \frac{\pi}{4},$$

which implies (20). The proof of (21) is similar.

Theorem 12.

$$M_k(x) = C_k x + O(x^{1/2}\log^{3+4(k-1)/3}x), \quad (22)$$

$$M_{*k}(x) = C_{*k} x + O(x^{1/2}\log^{3+2(k^2+k-2)/3}x), \quad (23)$$

where C_k and C_{*k} were defined in (20) and (21).

Proof. By Perron formula and by (10) for $c = 1+1/\log x$, $\log T \asymp \log x$ we have

$$M_k(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_k(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

Suppose $d = 1/2 - 1/\log x$. Let us shift the interval of integration to $[d-iT, d+iT]$. To do this consider an integral about a closed rectangle path with vertexes in $d-iT$, $d+iT$, $c+iT$ and $c-iT$. There are two poles in $s=1$ and $s=1/2$ inside the contour. The residue at $s=1$ was calculated in (19). The residue at $s=1/2$ is equal to $Dx^{1/2}$, *D* is constant, and will be absorbed by error term (see below).

Identity (12) implies

$$F_k(s) = Z(s)Z^{k-1}(2s)L_k(s),$$

where $L_k(s)$ is regular for $\Re s > 1/3$, so for each $\varepsilon > 0$ it is uniformly bounded for $\Re s > 1/3+\varepsilon$.

Let us estimate the error term using Lemma **5** and Lemma **6**. The error term absorbs values of integrals about three sides of the integration's rectangle. We take into account $Z(s) = \zeta(s)L(s,\chi_4)$. On the horizontal segments we have

$$\int_{d+iT}^{c+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll$$

$$\ll \max_{\sigma\in[d,c]} Z(\sigma+iT)Z^{k-1}(2\sigma+2iT)x^{\sigma}T^{-1} \ll$$

$$\ll x^{1/2}T^{2\theta-1}\log^{4(k-1)/3}T + xT^{-1}\log^{4/3}T,$$

It is well-known that $\zeta(s) \sim (s-1)^{-1}$ in the neighborhood of $s=1$. So on the vertical segment we have the following estimates. Near pole one can calculate that

$$\int_d^{d+i} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll x^{1/2}\int_0^1 \zeta^{k-1}(2d+2it)dt \ll$$

$$\ll x^{1/2}\int_0^1 \frac{dt}{|it-1/\log x|^{k-1}} \ll x^{1/2}\log^{k-1}x,$$

and for the rest of the vertical segment we get

$$\int_{d+i}^{d+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds \ll$$

$$\ll \left(\int_1^T |\zeta(1/2+it)|^4\frac{dt}{t}\int_1^T |L(1/2+it,\chi_4)|^4\frac{dt}{t}\right)^{1/4} \times$$

$$\times\left(\int_1^T |Z(1+2it)|^{2(k-1)}\frac{dt}{t}\right)^{1/2} \ll$$

$$\ll x^{1/2}\left(\log^5 T\cdot\log^{8(k-1)/3+1}T\right)^{1/2} \ll$$

$$\ll x^{1/2}\log^{3+4(k-1)/3}T.$$

The choice $T = x^{1/2+\varepsilon}$ finishes the proof of (22).

The proof of (23) is similar, but due to (13) one have replace $k-1$ by $(k^2+k-2)/2$.

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