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APPROXIMATION IN L^p AND THE POMPEIU PROPERTY

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Let H be the open upper half-space in R^n, $n \geq 2$, and assume that A is a non-empty, open, bounded subset of R^n such that $A \subset H$ and the exterior of A is connected. Let $p \in [2,+\infty)$. It is proved that there is a nonzero function with zero integrals over all sets in R^n congruent to A if and only if the indicator function of A is the limit in $L^p(H)$ of a sequence of linear combinations of indicator functions of balls in H with radii proportional to positive zeros of the Bessel function $J_{n/2}$. The proportionality coefficient here is the same for all balls and depends only on A. As an application some results on approximation in L^p are established.

KEY WORDS: Pompeu property, Lebesgue spaces, approximation.

ПРИБЛИЖЕНИЕ В L^p И СВОЙСТВО POMPEIU

Очаковская О.А.

Пусть H – открытое полупространство в R^n, $n \geq 2$, пусть A – непустое открытое ограниченное подмножество R^n, такое что $A \subset H$, а внешняя часть A связная, и пусть $p \in [2,+\infty)$. В работе доказано, что ненулевая функция с нулевыми интегралами по всем множествам в R^n конгруэнтная к A существует тогда и только тогда, если характеристическая функция A есть предел в $L^p(H)$ последовательности линейных комбинаций характеристических функций сфер в H с радиусом, пропорциональным положительным корням функции Бесселя $J_{n/2}$. В этом случае коэффициент пропорциональности один и тот же для всех сфер и зависит только от A. Приведены также некоторые результаты аппроксимации в L^p.

КЛЮЧЕВЫЕ СЛОВА: свойство Pompeu, пространства Лебега, аппроксимация.

НАБЛИЖЕННЯ В L^p ТА ВЛАСТИВІСТЬ POMPEIU

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Припустімо H – відкритий півпростір в R^n, $n \geq 2$, A – непуста відкрита обмежена підмножина R^n, така що $A \subset H$, а зовнішнья частина A зв'язна, а $p \in [2,+\infty)$. В роботі доведено, що ненульова функція з нульовими інтегралами по всім множинам в R^n конгруентна до A, існує тоді і тільки тоді, коли характеристичесна функція A є границею в $L^p(H)$ послідовності лінійних комбінацій характеристичних функцій сфер в H з радіусом, пропорційним позитивним корням функції Беселя $J_{n/2}$. В таким випадку коефіцієнт пропорціональності тий самий для усіх сфер і залежить тільки од A. Наведені також деякі результати апроксимації в L^p.

КЛЮЧОВІ СЛОВА: властивость Pompeu, простори Лебега, апроксимація.

1. Introduction. Let R^n be a real Euclidean space of dimension $n \geq 2$ with Euclidean norm $|\cdot|$ and let $M(n)$ be the group of its rigid motions. A non-empty open bounded subset A of R^n is called a *Pompeiu set* if for function $f \in L_{loc}(R^n)$ the equality

$$\int_{gA} f(x)dx = 0 \qquad (1)$$

holding for all $g \in M(n)$ yields $f = 0$. In this case, one says also that A has the *Pompeiu property*.

This notion takes its name from the Rumanian mathematician Dimitrie Pompeiu, who was the first to consider equation (1). Next, one says that A fails to have the *weak Pompeiu property* if there is a nonzero solution f of equality (1) such that

$$\int_{R^n} |f(x)|(1+|x|)^{-\alpha} dx < +\infty \qquad (2)$$

for some $\alpha > 0$ depending on f.

In the sequel, we write χ_A for the indicator function of A. Also let $\mathrm{Ext}(A)$ be the exterior of A (i.e., $\mathrm{Ext}(A) = R^n \setminus \overline{A}$ where $\overline{A}$ is the closure of A. For $\lambda > 0$, let $N_\lambda = \{r > 0 : J_{n/2}(r\lambda) = 0\}$ where J_k is the *k-th* order Bessel function of the first kind.

The classical Pompeiu problem about functions satisfying (1) has been studied by many authors (see the survey papers [1], [2], which contain an extensive bibliography; see also [3], [4]). There exists the long-standing conjecture that a ball is the only set among those whose boundary is homeomorphic to a sphere, which does not possess the Pompeiu property. A large amount of research has gone into this problem but it is still open. It is known that above stating conjecture is valid for the case where the boundary of A is locally the graph of a Lipschitz function and the function f in (1) is not real-analytic (see [4]).

Extremely interesting are local versions of the Pompeiu problem, when a function f is defined on a bounded domain G in R^n and relation (1) is required to hold only the set gA is contained in G. In this case the object is to determine conditions on the sets A and G under which (1) implies that $f = 0$ on G. The absence of the group structure provides a serious complicating factor (see [2–4]).

Of considerable interest is the case when G is a ball with radius $R > r(A)$ where $r(A)$ is the radius of the smallest closed ball containing the set A. One can in this case show that the Pompeiu property occurs when the size of the ball is sufficiently large compared with A (see [4]).

The following description of sets with the weak Pompeiu property has been obtained by V.V. Volchkov, see [5].

Theorem 1. *Let* A *be a non-empty, open, bounded subset of* R^n*.Then the following conditions are equivalent.*

(i) A *fails to have the weak Pompeiu property.*

(ii) There exists $\lambda = \lambda(A) > 0$ *such that the function* χ_A *is the limit of a sequence of linear combinations of the indicator functions of balls of radii* $r \in N_\lambda$ *convergent in* $L^1(R^n)$.

It is known that Theorem 1 is no longer valid for the space $L^p(R^n)$, $p \geq 2n/(n+1)$ instead of $L^1(R^n)$ (see [3, Part 2, Theorem 1.13]). The case $1 < p < 2n/(n+1)$ is still open.

We now formulate a similar result for the sets with the Pompeiu property obtaining by V.V. Volchkov (see [6]).

Theorem 2. *Assume that* A *is a non-empty, open, bounded subset in* R^n *such that* $\mathrm{Ext}(A)$ *is connected. Then the following items are equivalent.*

(i) A *is not a Pompeiu set.*

(ii) There exists $\lambda = \lambda(A) > 0$ *such that the function* χ_A *is the limit in* $L^1(R^n)$ *of a sequence of linear combinations of the indicator functions of balls with radii* $r \in N_\lambda$.

We point out that an analogue of this result remains to be true for the case where the Pompeiu property is considered for subsets of a symmetric space of non-compact type (see [6]).

In this paper, we obtain analogues of Theorems 1 and 2 in terms of approximation of χ_A in the space $L^p(H)$ where $2 \leq p < +\infty$ and

$$H = \{x = (x_1, ..., x_n) \in R^n : x_n > 0\}.$$

The proof of this result enables us to obtain some other results on approximation in L^p, $2 \leq p < +\infty$, by linear combinations of shifts.

2. Main results. We now proceed to the statement of the main results of the present paper. The first our result is as follows.

Theorem 3. *Let* A *be a non-empty, open, bounded subset of* R^n *such that* $\overline{A} \subset H$. *Let* $p \in [2, +\infty)$. *Then the following conditions are equivalent.*

(i) A *does not have the weak Pompeiu property.*

(ii) There exists $\lambda = \lambda(A) > 0$ *such that the function* χ_A *is the limit of a sequence of linear combinations of the indicator functions of balls in* H *of radii* $r \in N_\lambda$ *convergent in* $L^p(H)$.

In the sequel, we write $\hat{g}$ for the Fourier transform of integrable function (or distribution) in R^n. In addition, for $t > 0$ we set

$$H_t = \{x = (x_1, ..., x_n) \in R^n : x_n > t\}$$

Let G be a non-empty open subset of R^n. We denote by $E'(G)$ the space of all compactly supported distributions on G. For $R > 0$, we denote by B_R the open ball in R^n with radius R centered at the origin.

We write also $f*g$ for the convolution of functions f and g in the case where it there exists.

Using the method of the proof of Theorem 3 we can obtain the following statement which is an analogue of well-known Wiener's theorem on approximation by shifts in the space $L(R^n)$.

Theorem 4. *Let* $p\in[2,+\infty)$, $f\in L^p(H)$, *and assume that the support of function* f *is contained in* H_t *for some* $t>0$. *Suppose also that* $g\in L(R^n)$ *is supported in the ball* B_t *and that the zero set of* $\widehat{g}$ *contains some sphere with the center at the origin. Then the function* $f*g$ *is the limit in* $L^p(H)$ *of a sequence of linear combinations of indicator functions of balls in* H *with radii* $r\in N_\lambda$.

We point out that the condition on zero set of $\widehat{g}$ in previous theorem possesses for a broad class of functions g. For instance, it is valid for the case where g is a radial function.

The proof of Theorem 3 enables us to establish the following result providing a similar description for subsets in half-space with the Pompeiu property.

Theorem 5. *Suppose that* A *is a non-empty, open, bounded subset of* R^n *such that* $\overline{A}\subset H$ *and the set* $\mathrm{Ext}(A)$ *is connected. Let* $p\in[2,+\infty)$. *Then the following conditions are equivalent.*

(i) A *does not have the Pompeiu property.*

(ii) There exists $\lambda=\lambda(A)>0$ *such that the function* χ_A *is the limit in* $L^p(H)$ *of a sequence of linear combinations of the indicator functions of balls in* H *of radii* $r\in N_\lambda$.

As usual, the symbol Δ denotes the Laplace operator in R^n, that is, $\Delta=\dfrac{\partial^2}{\partial x_1^2}+\ldots+\dfrac{\partial^2}{\partial x_n^2}$.

The following result is an analogue of Theorem 4 for the case where g is a special distribution supported at the origin.

Theorem 6. *Let* $f\in L^p(H)$ *for some* $p\in[2,+\infty)$ *and assume that* $f=\Delta u+\lambda^2 u$ *for some* $u\in E'(H)$, $\lambda>0$. *Then* f *is the limit in* $L^p(H)$ *of a sequence of linear combinations of indicator functions of balls in* H *with radii* $r\in N_\lambda$.

Next, for $\lambda>0$ and $r\in N_\lambda$, we set

$$g_{\lambda,r}(x)=\begin{cases}\dfrac{J_{(n-2)/2}(\lambda|x|)\,r^{(n-2)/2}}{|x|^{(n-2)/2}\,J_{(n-2)/2}(\lambda r)}-1 & \text{if } |x|\le r\\[2ex] 0 & \text{if } |x|>r.\end{cases}$$

Assume now that $\zeta>0$. We require the following definition which is borrowed from [3, Part 1].

A non-empty open set $\Omega\subset R^n$ is called $\zeta-$domain if the following conditions hold:

(i) each point of Ω belongs to a closed ball of radius ζ contained in Ω;

(ii) the set of centers of all closed balls of radius ζ contained in Ω is connected.

We now state a result on the density in L^p of some system of shifts of radial function.

Theorem 7. *Let* $\lambda>0$ *be fixed. Suppose that* Ω *contains the half-space* H *and is an* $\zeta-$*domain for some* $\zeta\in N_\lambda$. *Then the set of all linear combinations of functions* $g_{\lambda,r}(x-h)$ *with* $r\in N_\lambda$ *and* $h\in H_r$ *is dense in* $L^p(\Omega)$ *for each* $p\in[2,+\infty)$.

The question about analogues of Theorems 3–7 for the case $1\le p<2$ remains open. Nobody knows is it possible to take in these theorems other domain instead of half-space.

To conclude we note that L^p-analogues of Wieners theorem on a bounded subset in R^n were considered in [3, Part 5, Chapter 2]. For other results relating to approximation by linear combinations of shifts, see [3, Part 5] containing an extensive bibliography, and also [4]. Some properties of functions satisfying (1) for the case where A is a ball were considered in [3, Part 2] and [7]. The local Pompeiu property for a family of compactly supported distributions was studied in [3] and [4].

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