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GENERALIZED SET-VALUED DIFFERENTIAL EQUATIONS

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In this paper the generalized set-valued differential equations with generalized derivative are considered and the existence and uniqueness theorems are proved.

KEY WORDS: generalized set-valued differential equations, theorems of existence and uniqueness, generalized derivative.

ОБОБЩЕННЫЕ МНОГОЗНАЧНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ

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В работе рассматриваются обобщенные многозначные дифференциальные уравнения с обобщенной производной. Доказаны соответстующие теоремы существования и единственности.

КЛЮЧЕВЫЕ СЛОВА: обобщенные многозначные дифференциальные уравнения, теоремы существования и единственности, обобщенная производная.

УЗАГАЛЬНЕНІ БАГАТОЗНАЧНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ

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В роботі розглядаються узагальнені багатозначні диференціальні рівняння з узагальненою похідною. Доведені відповідні теореми існування та єдиності.

КЛЮЧОВІ СЛОВА: узагальнені багатозначні диференціальні рівняння, теореми існування та єдиності, узагальнена похідна.

1. Introduction. The concept of derivative for set-valued mapping was first entered by M. Hukuhara [1]. Then the problems of differentiability of fuzzy mappings were considered by T. F. Bridgland [2], J.N. Tyurin [3], H.T. Banks and M.Q. Jacobs [4], A.V. Plotnikov [5, 6], A.N. Vityuk [7], B. Bede and S.G. Gal [8], A.V. Plotnikov and N.V. Skripnik [9]. The properties of these derivatives were considered in [10–18].

F.S. de Blasi and F. Iervolino begun studying of set-valued differential equations (SDEs) in semilinear metric spaces [12,19–21]. Now it developed in the theory of SDEs as an independent discipline. The properties of solutions, the impulsive SDEs, control systems and asymptotic methods for SDEs were considered [5,6,9–11,16–24]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used as an auxiliary tool to prove the existence results for differential inclusions. Also, one can employ SDEs in the investigation of fuzzy differential equations. Moreover, SDEs are a natural generalization of usual ordinary differential equations in finite (or infinite) dimensional Banach spaces [19].

In [9] a new concept of a derivative of a set-valued mapping that generalizes the concept of Hukuhara derivative was entered and a new type of a set-valued differential equation such that the diameter of its solution can whether increase or decrease (for example, to be periodic) was considered. In the ideological sense this definition of the derivative is close to the definitions proposed in [5,6,8].

In this paper the generalized set-valued differential equations with generalized derivative are considered and the existence and uniqueness theorems are proved.

2. Generalized differential equations with generalized derivative. Let $\mathrm{conv}(R^n)$ be a space of all nonempty convex closed sets of R^n with Hausdorff metric

$$h(A,B)=\min\{r\ge 0: A\subset B+S_r(0),\ B\subset A+S_r(0)\},$$

where $A,B\in\mathrm{conv}(R^n)$, $S_r(0)=\{s\in R^n:\|s\|\le r\}$.

Definition 1 [1]. Let $X,Y\in\mathrm{conv}(R^n)$. A set $Z\in\mathrm{conv}(R^n)$ such that $X=Y+Z$ is called a Hukuhara difference of the sets X and Y and is denoted by $X\overset{h}{-}Y$.

From Rådström's Embedding Lemma [25] it follows that if this difference exists, then it is unique.

First consider a differential equation with the generalized derivative that is similar to a differential equation with the Hukuhara derivative, i.e.

$$DX = F(t,X),\ X(t_0) = X_0, \qquad (1)$$

where DX is the generalized derivative of a set-valued mapping $X:[t_0,T]\to conv(R^n)$ [9], $F:[t_0,T]\times conv(R^n)\to conv(R^n)$ is a set-valued mapping, $X_0 \in conv(R^n)$.

Definition 2. A set-valued mapping $X:[t_0,T]\to conv(R^n)$ is said to be a solution of differential equation (1) if it is absolutely continuous and satisfies (1) almost everywhere on $[t_0,T]$.

Remark 1. Unlike the case of differential equations with Hukuhara derivative, if a differential equation with generalized derivative (1) has a solution then there exists an infinite number of solutions irrespective of the conditions on the right-hand side of the equation (see [9]).

Therefore we will consider the other differential equation with the generalized derivative:

$$DX(t)\overset{h}{-}\Phi(-\varphi(t))F_1(t,X(t)) = \Phi(\varphi(t))F_2(t,X(t)), \qquad (2)$$
$$X(t_0) = X_0,$$

where $t\in[t_0,T]$; $X:[t_0,T]\to conv(R^n)$; $X_0\in conv(R^n)$; $F_1,F_2:[t_0,T]\times conv(R^n)\to conv(R^n)$ are set-valued mappings; $\varphi:[t_0,T]\to R$ is a continuous function; function $\Phi(\varphi)=\begin{cases}1, & \varphi>0,\\ 0 & \varphi\le 0.\end{cases}$

Definition 3. A set-valued mapping $X:[t_0,T]\to conv(R^n)$ is called a solution of differential equation (2) if it is continuous and on any subinterval $[\tau_1,\tau_2]\subset[t_0,T]$, where function $\varphi(t)$ of constant signs, satisfies the integral equation

$$X(t)+\int_{\tau_1}^{t}\Phi(-\varphi(s))F_1(s,X(s))ds = X(\tau_1)+ +\int_{\tau_1}^{t}\Phi(\varphi(s))F_2(s,X(s))ds.$$

If on the interval $[\tau_1,\tau_2]$ the function $\varphi(t)>0$, then $X(t)$ satisfies the integral equation $X(t)=X(\tau_1)+\int_{\tau_1}^{t}F_2(s,X(s))ds$ for $t\in[\tau_1,\tau_2]$ and $diamX(t)$ increases.

If on the interval $[\tau_1,\tau_2]$ the function $\varphi(t)<0$, then we have $X(t)+\int_{\tau_1}^{t}F_1(s,X(s))ds = X(\tau_1)$, i.e. $X(t)=X(\tau_1)\overset{h}{-}\int_{\tau_1}^{t}F_1(s,X(s))ds$ and $diamX(t)$ decreases.

If on the interval $[\tau_1,\tau_2]$ the function $\varphi(t)<0$, then we have $X(t)\equiv X(\tau_1)$.

So we can enter the other equivalent definition of a solution of equation (2).

Definition 4. A set-valued mapping $X:[t_0,T]\to conv(R^n)$ is called a solution of differential equation (2) if it is absolutely continuous, satisfies (2) almost everywhere on $[t_0,T]$ and

$$diamX(t)=\begin{cases}\text{increases if } \varphi(t)>0,\\ \text{is constant if } \varphi(t)=0,\\ \text{decreases if } \varphi(t)<0.\end{cases}$$

Remark 2. It is obvious that the mappings $F_1(t,X)$, $F_2(t,X)$ define only the speed of changing of the mapping $X(\cdot)$ in case of its "decrease"($F_1(t,X)$) or "increase"($F_2(t,X)$) and function $\varphi(\cdot)$ defines what will occur to $X(\cdot)$ ["decrease" or "increase"]. If $\varphi(t)<0$ irrespective of $F_1(t,X(t))$ and $F_2(t,X(t))$ the mapping $X(\cdot)$ will be constant.

Let $CC(R^n)$ $(n\ge 2)$ be a space of all nonempty strictly convex closed sets of R^n and all element of R^n [27].

The following theorem of existence of the solution of equation (2) for case $CC(R^n)$ holds:

Theorem 1. *Let the set-valued mappings* $F_1(t,X)$, $F_2(t,X)$ $: R\times CC(R^n)\to CC(R^n)$ *in the domain* $Q=\{(t,X)\in R\times CC(R^n): t\in[t_0,t_0+a],\ h(X,X_0)\le b\}$ *satisfy the following conditions:*

i) for any fixed X *the set-valued mappings* $F_1(\cdot,X)$, $F_2(\cdot,X)$ *are measurable;*

ii) for almost every fixed t *the set-valued mappings* $F_1(t,\cdot)$, $F_2(t,\cdot)$ *are continuous;*

iii) $|F_1(t,X)|\le m_1(t)$, $|F_2(t,X)|\le m_2(t)$, *where* , $m_1(\cdot)$ *are summable on* $t\in[t_0,t_0+a]$*;*

iv) $\varphi(t)$ *is continuous and has the finite number of intervals where* $sign(\varphi(t))=\pm1$*;*

v) $int X_0\ne\varnothing$.

Then there exists a solution of equation (2) defined on the interval $t\in[t_0,t_0+d]$*, where* $d>0$ *satisfies the conditions*

a) $d\le a$*;*

b) $\phi_i(t_0+d)\le b$, *where* $\phi_i(t)=\int_{t_0}^{t} m_i(s)ds$, $i=1,2$*;*

c) $\int_{\mu[t_0,t_0+d]} m_1(s)ds \le \frac{\theta}{2}$,

where $\theta=\min_{\|\psi\|=1}|C(X_0,\psi)+C(X_0,-\psi)|$, $C(X,\psi)=\max_{x\in X}(x_1\psi_1+...+x_n\psi_n)$, $\mu[t_0,t_0+d]\subset[t_0,t_0+d]:\varphi(t)<0$ *for* $t\in\mu[t_0,t_0+d]$.

Proof. Let us consider some cases.

1) $\varphi(t)>0$ for $t\in[t_0,t_0+a]$. Then equation (2) is the ordinary differential equation with Hukuhara derivative $D_HX=F_2(t,X)$, $X(t_0)=X_0$.

Therefore, using [17] we get that equation (2) has a solution $X(t)$ defined on $[t_0,t_0+d]$, where d satisfies the condition $d=\min\{a,\gamma\}$, $\int_{t_0}^{t_0+\gamma} m_2(s)ds=b$.

2) $\varphi(t)\equiv 0$ for $t\in[t_0,t_0+a]$. Then equation (2) is the ordinary differential equation with Hukuhara derivative $D_HX=\{0\}$, $X(t_0)=X_0$ and therefore, $X(t)\equiv X_0$ is solution of (2) on $[t_0,t_0+a]$.

3) $\varphi(t)<0$ for $t\in[t_0,t_0+a]$. Then equation (2) is the equation with the generalized derivative

$$DX(t)\overset{h}{-}F_1(t,X(t))=\{0\},\ X(t_0)=X_0. \qquad (3)$$

According to Definition 3 consider the following integral equation

$$X(t)=X_0\overset{h}{-}\int_{t_0}^{t}F_1(s,X(s))ds \qquad (4)$$

for $t\in[t_0,t_0+a]$ and prove the existence of solution on some interval $[t_0,t_0+d]$.

3a) As $|F_1(t,X)|\le m_1(t)$ for $(t,X)\in Q$, then $F_1(t,X)\subset S_{m_1(t)}(0)$, where $S_r(a)=\{x\in R^n:\|x-a\|\le r\}$.

So $\int_{t_0}^{t}F_1(s,X)ds\subset\int_{t_0}^{t}S_{m_1(s)}(0)ds=S_{\int_{t_0}^{t}m_1(s)ds}(0)$.

Define by $S(t)=S_{\int_{t_0}^{t}m_1(s)ds}(0)$. It is obviously, that if $t_0<t_1<t_2<t_0+a$, then $\{0\}=S(t_0)\subset S(t_1)\subset S(t_2)\subset S(t_0+a)$.

As $X_0\in CC(R^n)$ and $\operatorname{int}X_0\ne\varnothing$, then there exists $d_1>0$ such that the set $S(t)$ can be embedded in the set X_0 for all $t\in[t_0,t_0+d_1]$ (i.e. there exists $\zeta(t)$ such that $S(t)+\zeta(t)\subset X_0$) and is not embedded for $t>t_0+d_1$. And, it is obviously, that d_1 can be found out from the equation $\int_{t_0}^{t_0+d_1}m_1(s)ds=\frac{\theta}{2}$.

Therefore, for all

$$(t,X)\in Q_1=\{(t,X)\in R\times CC(R^n):$$
$$:t\in[t_0,t_0+d_1],h(X,X_0)\le b\}$$

the set $\int_{t_0}^{t}F_1(s,X)ds$ is embedded in the set X_0.

3b) As $F_1(t,X)\in CC(R^n)$ for all $(t,X)\in Q_1$, then $\int_{t_0}^{t}F_1(t,X)ds\in CC(R^n)$ for all $(t,X)\in Q$ [27]. Therefore, as $X_0\in CC(R^n)$ and the set $S(t)$ can be embedded in the set X_0 for all $t\in[t_0,t_0+d_1]$, then the Hukuhara difference $X_0\overset{h}{-}\int_{t_0}^{t}F_1(s,X)ds$ exists for all $(t,X)\in Q_1$ [27].

3c) Let us find $d_2>0$ such that $\int_{t_0}^{t_0+d_2}m_1(s)ds=b$ and consider $d=\min\{a,d_1,d_2\}$.

3d) Choose any natural k. Sequentially on the intervals $t_0+i\Delta\le t\le t_0+(i+1)\Delta$, $\Delta=\frac{d}{k}$, $i=0,...,k-1$ let us build the successive approximations of the solution

$$X^k(t)=X_0 \text{ for } t_0-\Delta\le t\le t_0,$$

$$X^k(t)=X_0\overset{h}{-}\int_{t_0}^{t}F_1(s,X^k(s-\Delta))ds \text{ for } t\in[t_0,t_0+d].$$

By 3b) $X^k(t)$ exist and $X^k(t)\in CC(R^n)$ for all $k\in N$ and $t\in[t_0,t_0+d]$. Also by conditions i) and ii) of the theorem $X^k(t)$ is continuous on $[t_0,t_0+d]$ for all $k\in N$.

Besides

$$h(X^k(t),X_0)=h\left(X_0\overset{h}{-}\int_{t_0}^{t}F_1(s,X^k(s-\Delta))ds,X_0\right)\le$$

$$\le\int_{t_0}^{t}h(F_1(s,X^k(s-\Delta)),\{0\})$$

$$\le\int_{t_0}^{t}m_1(s)ds\le\phi_1(t_0+d)\le b.$$

Hence, the sequence of the set-valued mappings $\{X^k(t)\}_{k=1}^{\infty}$ in uniformly bounded: $h(X^k(t),\{0\}) \le h(X_0,\{0\}) + b$.

Let us show that the set-valued mappings $X^k(t)$ are equicontinuous. For any $\alpha < \beta$ $\alpha, \beta \in [t_0, t_0 + d]$ and any natural k the inequality holds

$$h(X^k(\alpha), X^k(\beta)) =$$

$$h\left(X_0 \frac{h}{-} \int_{t_0}^{\alpha} F_1(s, X^k(s-\Delta))ds, X_0 \frac{h}{-} \int_{t_0}^{\beta} F_1(s, X^k(s-\Delta))ds \right) =$$

$$\le \int_{\alpha}^{\beta} h(F_1(s, X^k(s-\Delta)),\{0\})ds \le \int_{\alpha}^{\beta} m_1(s)ds = \phi_1(\beta) - \phi_1(\alpha)$$

The function $\phi_1(t)$ is absolutely continuous on $[t_0, t_0 + d]$ as the integral of the summable function with a variable top limit. Hence, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all α, β such that $0 \le \beta - \alpha < \delta$ the inequality $h(X^k(\alpha), X^k(\beta)) < \varepsilon$ is fair, the sequence $\{X^k(t)\}_{k=1}^{\infty}$ is equicontinuous.

According to Askoli theorem [28] we can choose a uniformly converging subsequence of the sequence $\{X^k(t)\}_{k=1}^{\infty}$. Its limit is a continuous set-valued mapping that we will denote by $X(t)$. As

$$h(X^k(s-\Delta), X(s)) \le h(X^k(s-\Delta), X^k(s)) + \\ + h(X^k(s), X(s))$$

and the first summand is less than ε for $\Delta = \frac{d}{k} < \delta$ in view of the equicontinuity of the set-valued mappings $\{X^k(t)\}_{k=1}^{\infty}$, then along the chosen subsequence $\{X^k(s-\Delta)\}_{k=1}^{\infty}$ converges to $X(t)$. Owing to the theorem conditions in (3) it is possible to pass to the limit under the sign of the integral. We receive that the set-valued mapping $X(t)$ satisfies equation (4) and $X(t_0) = X_0$, i.e. $X(t)$ is the solution of (3) on the interval $[t_0, t_0 + d]$.

4) In case when the function $\varphi(t)$ changes sign on the interval $[t_0, t_0 + a]$, the existence of the solution is proved combining cases 1)–3). The theorem is proved.

Theorem 2. *Let the set-valued mappings* $F_1(t, X)$, $F_2(t, X)$ $R \times CC(R^n) \to CC(R^n)$ *satisfy the conditions of Theorem 1 and satisfy the conditions*

$$h(F_1(t, X'), F_1(t, X'')) \le L_1 h(X', X''),$$
$$h(F_2(t, X'), F_2(t, X'')) \le L_2 h(X', X'')$$

for all $(t, X'), (t, X'') \in Q$. *Then there exists the unique solution of equation (2) defined on the interval* $t \in [t_0, t_0 + d]$.

The proof is similar to [17,24].

Remark 3. Also it is possible to prove the similar results if $CC(R^n)$ be a space of all nonempty M-strongly convex closed sets of R^n and all element of R^n [29].

3. Conclusions. In this paper the concept of generalized differentiability (proposed in [9]) for set-valued mappings is used. The new type of the set-valued differential equation – generalized set differential equations – is considered. The existence and uniqueness theorems for set-valued differential equations with generalized derivative are proved.

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