UDK 519.21

CONDITIONS OF EXISTENCE WITH PROBABILITY ONE OF GENERALIZED SOLUTION FOR THE BOUNDARY-VALUE PROBLEM OF RECTANGULAR MEMBRANE'S VIBRATION WITH RANDOM INITIAL CONDITIONS

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The first boundary problem for the hyperbolic type equation of the mathematics physics under random initial conditions is analyzed in the article, specifically the equation of rectangular membrane's vibration. The conditions of existence with probability one of generalized solution are investigated. Estimation of the distribution of supremum solutions of such sort of problems has been obtained.

KEY WORDS: hyperbolic type equation, random initial conditions, rectangular membrane vibrations.

УСЛОВИЯ ВЕРОЯТНОСТНОГО СУЩЕСТВОВАНИЯ ОБОБЩЕННОГО РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ КОЛЕБАНИЙ ПРЯМОУГОЛЬНОЙ МЕМБРАНЫ СО СЛУЧАЙНЫМИ НАЧАЛЬНЫМИ УСЛОВИЯМИ

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В статье рассматривается первая краевая задача для уравнения математической физики гиперболического типа при случайных начальных условиях, в частности, уравнение колебаний прямоугольного мембраны. Исследованы условия вероятностного существования одного из обобщенных решений. Получена оценка для распределения граничных решений такого рода задач.

КЛЮЧЕВЫЕ СЛОВА: уравнение гиперболического типа, случайные начальные условия, колебания прямоугольной мембраны.

УМОВИ ІМОВІРНІСНОГО ІСНУВАННЯ УЗАГАЛЬНЕНОГО РОЗВ'ЯЗКУ КРАЙОВИХ ЗАДАЧ КОЛИВАНЬ ПРЯМОКУТНОЇ МЕМБРАНИ З ВИПАДКОВИМИ ПОЧАТКОВИМИ УМОВАМИ

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В роботі розглядається перша крайова задача для рівняння математичної фізики гіперболічного типу при випадкових початкових умовах, зокрема, рівняння коливань прямокутної мембрани. Досліджені умови імовірнісного існування одного з узагальнених рішень. Отримана оцінка для розподілу граничних рішень такого роду задач.

КЛЮЧОВІ СЛОВА: рівняння гіперболічного типу, випадкові початкові умови, коливання прямокутної мембрани.

1. Introduction. The influence of random factors should often be taken into account in solving problems of mathematical physics. These factors can be of a diverse nature: random boundary conditions and random initial conditions, random forces acting on the system, random coefficients of differential operators, etc. This brings up the necessity of analyzing specific features of the problem in question. The keypoints usually are: existence and uniqueness of the solution, the possibility of a constructive approximation of the solution and type of convergence of approximating functions to the solution, behavior of different functional of the solution, etc. Different methods of investigation are applied depending on the type of the problem, specifics of random factors involved and the questions to be studied.

We consider a boundary problem of membrane oscillations with random strongly Orlicz initial conditions.

The first boundary problem for the hyperbolic type equation of the mathematics physics under random initial conditions is analyzed in the article, specifically the equation of rectangular membrane's vibration. The conditions of existence with probability one of generalized solution are investigated. For such a problem has been got the estimation for the distribution of supremum solving.

Similar problems for the hyperbolic type equations are considered in [1, 2, 3], parabolic type equations are considered in [4]. Further references can be found in [5].

Results of this work can be used to justify the use of the Fourier method in terms of the correlation functions, for study the rate of convergence of series of functions, representing the solution of the problem, for model making, etc.

2. Formulation of the problem. Consider the problem of the rectangular membrane [6]:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial t^2}=0, \quad (1)$$

$$x\in[0,a],\ y\in[0,b],\ t\in[0,\tau],$$

$$u\big|_{t=0}=\xi(x,y), \quad \frac{\partial u}{\partial t}\bigg|_{t=0}=\eta(x,y), \quad (2)$$

$$u\big|_S=0, \quad (3)$$

where u is the deviation of the membrane from its equilibrium position, which coincides with the plane x, y and S is a boundary of rectangle $0<x<a,\ 0<y<b$.

Let initial conditions $(\xi(x,y), x\in[0,a], y\in[0,b])$ and $(\eta(x,y), x\in[0,a], y\in[0,b])$ are strongly Orlicz stochastic processes [7] defined on a common complete probability space $(\Omega,\Im,P)$, and such that

$$\xi(0,y)=\xi(x,0)=\xi(a,y)=\xi(x,b)=0,$$
$$\eta(0,y)=\eta(x,0)=\eta(a,y)=\eta(x,b)=0$$

almost surely.

When solving similar problems (1)–(3) by using the Fourier method regardless of whether initial conditions are random or nonrandom, we look for a solution of the form

$$u(t,x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}\,t+\frac{b_{nm}}{\sqrt{\lambda_{nm}}}\sin\sqrt{\lambda_{nm}}\,t\right], \quad (4)$$

where $a_{nm}=\int_0^a\int_0^b \xi(x,y)V_{nm}(x,y)dxdy$,

$b_{nm}=\int_0^a\int_0^b \eta(x,y)V_{nm}(x,y)dxdy$, λ_{nm} and V_{nm} are eigenvalues and eigenfunctions of the following Sturm-Liouville problem [6]: $V_{xx}+V_{yy}+\lambda V=0$, $V\big|_S=0$, where λ_{nm} and V_{nm} have the following forms

$$V_{nm}(x,y)=\frac{2}{\sqrt{ab}}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y,$$

$$\lambda_{nm}=\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right).$$

Let $D=[0,a]\times[0,b]\times[0,\tau]$.

Definition1. The solution (4) is called a generalized solution of problem (1)–(3) in the domain D if series (4) converge uniformly in probability.

3. Main result. Let us consider the conditions that ensure the uniform convergence in probability seriesr (4) if the initial conditions are random processes from Orlicz space.

Lemma 1 ([3]). *Let initial conditions* $(\xi(x,y),\ x\in[0,a],\ y\in[0,b])$ *and* $(\eta(x,y),\ x\in[0,a],\ y\in[0,b])$ *be strongly Orlicz stochastic processes and assume that the series (4) converge uniformly in probability. Then the random series (4) also is a strongly Orlicz stochastic process.*

For $N\ge 1$ put

$$S_N(t,x,y)=\sum_{n=1}^{N}\sum_{m=1}^{N}V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}\,t+\frac{b_{nm}}{\sqrt{\lambda_{nm}}}\sin\sqrt{\lambda_{nm}}\,t\right].$$

Theorem 1. *Let* $(\xi(x,y),\ x\in[0,a],\ y\in[0,b])$, $(\eta(x,y),\ x\in[0,a],\ y\in[0,b])$ *be strongly Orlicz stochastic processes. In order that a generalized solution of the problem (1)–(3) exists with probability one in the domain* D *and can be represented in the form of series (4) it is sufficient that:*

1) *for all* $(t,x,y)\in D$ *the series*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}V_{nm}(x,y)V_{kl}(x,y)\Big[Ea_{nm}a_{kl}\times\cos\sqrt{\lambda_{nm}}\,t\cos\sqrt{\lambda_{kl}}\,t+\frac{Eb_{nm}b_{kl}}{\sqrt{\lambda_{nm}\lambda_{kl}}}\sin\sqrt{\lambda_{nm}}\,t\sin\sqrt{\lambda_{kl}}\,t+2\frac{Ea_{nm}b_{kl}}{\sqrt{\lambda_{kl}}}\cos s\sqrt{\lambda_{nm}}\,t\sin\sqrt{\lambda_{kl}}\,t\Big]$$

converges;

2) *for* $N\ge 1$

$$\sup_{\substack{|x-x_1|\le h\\|y-y_1|\le h\\|t-t_1|\le h}}\left(E\left|S_N(t,x,y)-S_N(t_1,x_1,y_1,t_1)\right|^2\right)^{\frac{1}{2}}\le\sigma(h),$$

where $\sigma(h)$ *is a monotone increasing continuous function such that* $\sigma(h)\to 0$ *as* $h\to 0$, *moreover*

$$\int_0^{\varepsilon}U^{(-1)}\left[\left(\frac{a}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\times\left(\frac{\tau}{2\sigma^{(-1)}(u)}+1\right)\right]du<\infty \quad (5)$$

where $\sigma^{(-1)}(h)$ *is the inverse function to* $\sigma(h)$.

Proof. Condition 1) implies that the series (4) converge in the mean square sense. According to theorem 4.1 of the paper [4] and Lemma 1 series (4) converges in probability space $C(D)$.

Example. Assume that $(\xi(x,y),\ x\in[0,a],\ y\in[0,b])$, $(\eta(x,y),\ x\in[0,a],\ y\in[0,b])$ are strongly Orlicz processes from the $L_u(\Omega)$. Let $u(x)$ be a function such that $u(x)=|x|^p$ for some $p>3$ and all $|x|>1$. Then conditions (5) of Theorem 1 holds the function $\sigma(h)=C|h|^{\delta}$ for $0<\delta\le 1$. Indeed for $\varepsilon>0$

$$I=\int_{0+}^{\varepsilon}\left(\left(\frac{aC^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}+1\right)\cdot\left(\frac{bC^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}+1\right)\cdot\left(\frac{\tau C^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}+1\right)\right)^{\frac{1}{p}}du,$$

$$I\le\int_{0+}^{\varepsilon}\left(\frac{aC^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}\cdot\frac{bC^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}\cdot\frac{\tau C^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}}\right)^{\frac{1}{p}}du=D\int_{0+}^{\varepsilon}\left(\frac{1}{u^{\frac{3}{p\delta}}}\right)du,$$

where $D=\dfrac{(ab\tau)^{1/p}C^{3/p\delta}}{2^{3/p}}$. The latter integral converges under $\delta>\dfrac{3}{p}$.

Theorem 2. *Let* $(\xi(x,y),\ x\in[0,a],\ y\in[0,b])$, $(\eta(x,y),\ x\in[0,a],\ y\in[0,b])$ *be strongly Orlicz stochastic processes from the* $L_u(\Omega)$, *where* $u(x)=|x|^p$, *for some* $p>3$ *and all* $|x|>1$. *In order that a generalized solution of problem (1)–(3) exists in the domain* D *and can be represented in the form of series (4) it is sufficient that:*

1) *the series*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[|Ea_{nm}a_{kl}|+\frac{|Eb_{nm}b_{kl}|}{\sqrt{\lambda_{nm}\lambda_{kl}}}+2\frac{|Ea_{nm}b_{kl}|}{\sqrt{\lambda_{kl}}}\right]$$

converges;

2) $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[(Ea_{nm}^2)^{\frac{1}{2}}+(Eb_{nm}^2)^{\frac{1}{2}}\right](n^{\delta}+m^{\delta})<\infty$ *for arbitrary* $\delta>\dfrac{3}{p}$.

Proof. Condition 1) of this theorem implies condition 1) of theorem 1. Example 1 and lemma 1 imply that conditions 2) of the theorem 1 follows from condition 2) of the theorem 2. It is clear that

$$\left(E|S_N(t,x,y)-S_N(t_1,x_1,y_1)|^2\right)^{\frac{1}{2}}=$$

$$E\left(\left|\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{2}{\sqrt{ab}}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\left[a_{nm}\cos\sqrt{\lambda_{nm}}t+\frac{b_{nm}}{\sqrt{\lambda_{nm}}}\sin\sqrt{\lambda_{nm}}t\right]-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{2}{\sqrt{ab}}\sin\frac{n\pi}{a}x_1\times\right.\right.$$

$$\left.\left.\times\sin\frac{m\pi}{b}y_1\left[a_{nm}\cos\sqrt{\lambda_{nm}}t_1+\frac{b_{nm}}{\sqrt{\lambda_{nm}}}\sin\sqrt{\lambda_{nm}}t_1\right]\right|^2\right)^{\frac{1}{2}}\le$$

$$\le\frac{2}{\sqrt{ab}}\sum_{n=1}^{N}\sum_{m=1}^{N}\left[\left(Ea_{nm}^2\right)^{\frac{1}{2}}\left|\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\cos\sqrt{\lambda_{nm}}t-\sin\frac{n\pi}{a}x_1\sin\frac{m\pi}{b}y_1\cos\sqrt{\lambda_{nm}}t_1\right|+\frac{(Eb_{nm}^2)^{\frac{1}{2}}}{\sqrt{\lambda_{nm}}}\left|\sin\frac{n\pi}{a}x\times\sin\frac{m\pi}{b}y\sin\sqrt{\lambda_{nm}}t-\sin\frac{n\pi}{a}x_1\sin\frac{m\pi}{b}y_1\sin\sqrt{\lambda_{nm}}t_1\right|\right].$$

Further

$$\left|\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\sin\sqrt{\lambda_{nm}}t-\sin\frac{n\pi}{a}x_1\times\times\sin\frac{m\pi}{b}y_1\sin\sqrt{\lambda_{nm}}t_1\right|\le 2\left[\left|\sin\frac{n\pi(x-x_1)}{2a}\right|+\left|\sin\frac{m\pi(y-y_1)}{2b}\right|+\left|\cos\frac{\sqrt{\lambda_{nm}}(t-t_1)}{2}\right|\right].\tag{6}$$

The inequality

$$|\sin\alpha|\le|\alpha|^{\delta},\ 0<\delta\le 1$$

together with (6) implies that

$$\left|\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\cos\sqrt{\lambda_{nm}}t-\sin\frac{n\pi}{a}x_1\times\times\sin\frac{m\pi}{b}y_1\cos\sqrt{\lambda_{nm}}t_1\right|\le 2\left[\left|\frac{n\pi(x-x_1)}{2a}\right|^{\delta}+\left|\frac{m\pi(y-y_1)}{2b}\right|^{\delta}+\left|\frac{\sqrt{\lambda_{nm}}(t-t_1)}{2}\right|^{\delta}\right]\le 2^{2-\delta}\pi^{\delta}h^{\delta}(n^{\delta}+m^{\delta}).$$

Similarly

$$\left|\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\sin\sqrt{\lambda_{nm}}t-\sin\frac{n\pi}{a}x_1\right.$$
$$\left.\times\sin\frac{m\pi}{b}y_1\sin\sqrt{\lambda_{nm}}t_1\right|\le 2^{2-\delta}\pi^{\delta}h^{\delta}(n^{\delta}+m^{\delta}).$$

One can easily obtain that

$$\left(E\left|S_N(t,x,y)-S_N(t_1,x_1,y_1)\right|^2\right)^{\frac{1}{2}}\le C|h|^{\delta},$$

where

$$C=\frac{2^{2-\delta}\pi^{\delta}}{\sqrt{ab}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left|Ea_{nm}^2\right|^{\frac{1}{2}}+\frac{\left|Eb_{nm}^2\right|^{\frac{1}{2}}}{\sqrt{\lambda_{nm}}}\right](n^{\delta}+m^{\delta}).$$

4. Estimates of the distribution of the supremum of a solution of the boundary value problem.

Theorem 3 ([4]). *Let* (T,ρ) *be a compact metric space and* $N(u)$ *the metric massiveness of the space* (T,ρ), *that is, the minimum number of closed balls of radius* u *that cover* (T,ρ). *Let* $X=\{X(t),t\in T\}$ *be a separable stochastic process from the space* $L_U(\Omega)$ *and let the function* U *satisfy the* g*-condition. Assume that there exists a monotone increasing continuous function* $\sigma=\sigma(h),\sigma(0)=0,0\le h\le\sup_{t,s\in T}\rho(t,s)$ *such that*

$$\sup_{\rho(t,s)\le h}\|X(t)-X(s)\|_U\le\sigma(h).$$

If for some $\varepsilon>0$

$$\int_0^{\varepsilon}\chi_U\left(N(\sigma^{(-1)}(u))\right)du<\infty, \tag{7}$$

where

$$\chi_U(n)=\begin{cases}n, & n<U(z_0),\\ C_U U^{(-1)}(n), & n\ge U(z_0),\end{cases}$$
$$C_U=K(1+U(z_0))\max(1,A),$$

z_0,K,A *are constants from definition of C-function ([5]) and* $\sigma^{(-1)}(h)$ *is the inverse of* $\sigma(h)$, *then the random variable* $\sup_{t\in T}|X(t)|$ *belongs to the space* $L_U(\Omega)$ *with probability one and*

$$\left\|\sup_{t\in T}\right\|_U\le\|X(t_0)\|_U+\frac{1}{\theta(1-\theta)}\int_0^{\varpi_0\theta}\chi_U(N(\sigma^{(-1)}(u)))du=$$
$$=B(t_0,\theta),$$

where $t_0\in T,\varpi=\sigma(\sup_{t\in T}\rho(t_0,t)),0<\theta<1.$ *In addition, for all* $\varepsilon>0$ *the following inequality holds:*

$$P\left\{\sup_{t\in T}|X(t)|>\varepsilon\right\}\le\left(U\left(\frac{\varepsilon}{B(t_0,\theta)}\right)\right)^{-1}.$$

The next result follows from Theorem 3,

Theorem 4. *Let in conditions of theorem 3* $T=\{0\le x\le a,0\le y\le b,$ $0\le t\le\tau\}$ $\rho((x,y,t),(x_1,y_1,z_1))=\max\{|x-x_1|,$ $|y-y_1|,|t-t_1|\}$. *Then the condition (7) holds, when for all* $\varepsilon>0$

$$\int_0^{\varepsilon}U^{(-1)}\left[\left(\frac{a}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\times\right.$$
$$\left.\times\left(\frac{\tau}{2\sigma^{(-1)}(u)}+1\right)\right]du<\infty,$$

$$B(x_0,y_0,t_0,\theta)\le\tilde{B}(x_0,y_0,t_0,\theta)=$$
$$=\|X(x_0,y_0,t_0)\|_U+\frac{1}{\theta(1-\theta)}\times$$
$$\times\int_0^{\varpi_0\theta}U^{(-1)}\left[\left(\frac{a}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{\tau}{2\sigma^{(-1)}(u)}+1\right)\right]du.$$

In addition, for all $\varepsilon>0$ *the following inequality holds:*

$$P\left\{\sup_{t\in T}|X(t)|>\varepsilon\right\}\le\left(U\left(\frac{\varepsilon}{B(x_0,y_0,t_0,\theta)}\right)\right)^{-1}.$$

Put

$$u_N(x,y,t)=\sum_{n=N}^{\infty}\sum_{m=N}^{\infty}V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}t+\right.$$
$$\left.+\frac{b_{nm}}{\sqrt{\lambda_{nm}}}\sin\sqrt{\lambda_{nm}}t\right].$$

As in theorem 1, assume that

$$\sup_{\substack{|x-x_1|\le h\\|y-y_1|\le h\\|t-t_1|\le h}}\left(E\left|S_N(x,y,t)-S_N(x_1,y_1,t_1)\right|^2\right)^{\frac{1}{2}}\le\sigma(h),$$

where $\sigma(h)$ is a monotone increasing continuous function such that $\sigma(h)\to 0$ as $h\to 0$ moreover

$$\int_0^{\varepsilon}U^{(-1)}\left[\left(\frac{a}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\times\right.$$
$$\left.\times\left(\frac{\tau}{2\sigma^{(-1)}(u)}+1\right)\right]du<\infty,$$

where $\sigma^{(-1)}(h)$ is the inverse function to $\sigma(h)$. Then

$$P\left\{\sup_{t\in T}\left|u_N(x,y,t)\right| > \varepsilon\right\} \le \left(U\left(\frac{\varepsilon}{B(x_0,y_0,t_0,\theta)}\right)\right)^{-1},$$

where

$$B(x_0,y_0,t_0,\theta) \le \tilde{B}(x_0,y_0,t_0,\theta) =$$
$$= \left\|u(x_0,y_0,t_0)\right\|_U + \frac{1}{\theta(1-\theta)}\times$$
$$\times \int_0^{\varpi_0\theta} U^{(-1)}\left[\left(\frac{p}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{q}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{\tau}{2\sigma^{(-1)}(u)}+1\right)\right]du.$$

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