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A CHARACTERIZATION OF SOLUTIONS FOR SOME DIFFERENTIAL EQUATIONS IN TERMS OF WEIGHT MEANS OVER DISKS

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A description of solutions of some integral equations has been obtained. A two-radii theorem is obtained as well.

KEY WORDS: differential equations, integral mean values

ПРЕДСТАВЛЕНИЕ РЕШЕНИЙ НЕКОТОРЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ТЕРМИНАХ ВЗВЕШЕННЫХ СРЕДНИХ НА ДИСКАХ

Трофименко О.Д.

Получено описание решений некоторых дифференциальных уравнений, а также доказана теорема о двух радиусах.

КЛЮЧЕВЫЕ СЛОВА: дифференциальные уравнения, средние интегральные значения.

ПРЕДСТАВЛЕННЯ РІШЕНЬ ДЕЯКИХ ДИФФЕРЕНЦІАЛЬНИХ РІВНЯНЬ В ТЕРМІНАХ ЗВАЖЕНИХ СЕРЕДНІХ НА ДИСКАХ

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Отримано опис рішень деяких диференціальних рівнянь, а також доказана теорема о двох радіусах.

КЛЮЧОВІ СЛОВА: диференціальні рівняння, середні інтегральні значення.

1. Introduction. Characterization of solutions for differential equations in terms of various integral mean values has been studied by many authors (see [1–9] and references in these papers).

The classes of functions on subsets of the compact plane that satisfy the conditions of the next type is studied in this work

$$\sum_{n=s}^{m-1} \frac{r^{2n+2}}{2(n-s)!(n+1)!}\left(\frac{\partial}{\partial z}\right)^{n-s}\left(\frac{\partial}{\partial \bar{z}}\right)^{n} f(z) = =\frac{1}{2\pi}\iint_{|\varsigma-z|\le r} f(\varsigma)(\varsigma-z)^s d\xi d\eta, \qquad (1)$$

where $m \in N$ and $s \in 0,...,m-1$ are fixed. Also r is fixed or belongs to the set of two elements.

We point out that this equation holds for m-analytic functions (see [10]). The set of functions from C^{2m-2-s} in some domain, that satisfies (1) with all possible z and r is of great interest.

To find the form of such functions the question on uniqueness results under certain assumptions arises.

The main results of this work are as follows.

1) An uniqueness theorem for solutions of the mean value equation has been obtained (see Theorem 1).

A theorem which indicates an exactness of this uniqueness theorem, is obtained as well (see Theorem 2).

2) The description of all smooth solutions for (1) in a disk with radius $R > r$ with one fixed r is obtained (see Theorem 3 below);

3) The two-radii theorem is obtained. It turn out that this theorem characterizes class of solution for equation

$$\left(\frac{\partial}{\partial z}\right)^{m-s}\left(\frac{\partial}{\partial \bar{z}}\right)^{m} f = 0 \qquad (2)$$

in terms of equation (1) (see Theorem 4).

Note that the case $s \ge m$ that corresponds to the zero integral mean value in the right hand side of (1), has been studied in the works of L.Zalcman and V.V.Volchkov (see [3], [11,12]).

The first results that deal with the mean value theorem for polyanalytic functions, are contained in [13,14].

2. Main results. Let J_ν be the Bessel function of the first kind with index ν. For $\rho \geq 0$, $\lambda \in C$, $k \in Z$, let

$$\Phi_{\lambda,\eta,k}(\rho) = \left(\frac{d}{dz}\right)^{\eta} \left(J_k(z\rho)\right)\big|_{z=\lambda}.$$

Let also

$$g_r(z) = \frac{J_{s+1}(rz)}{(zr)^{s+1}} - \sum_{n=s}^{m-1} \frac{(zr)^{2(n-s)}(-1)^{n-s}}{(n+1)!(n-s)!2^{2n-s+1}},$$

and $Z(g_r) = \{z \in C : g_r(z) = 0\}$,

$Z_r = Z(g_r) \setminus \{z \in C : \arg z \in (-\pi, -\pi/2] \cup (\pi/2, \pi]\}$.

For $\lambda \in Z_r$, by the symbol n_λ we denote the multiplicity of zero λ of the entire function g_r.

Let $D_R = \{z \in C : |z| < R\}$.

To any function $f \in C(D_R)$ there are corresponds Fourier series

$$f(z) \sim \sum_{k=-\infty}^{\infty} f_k(\rho) e^{ik\phi},$$

where

$$f_k(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it}) e^{-ikt} dt,$$

and $0 \leq \rho < R$.

For functions with corresponding smoothness, an uniqueness theorem for integral equation (1) as follows.

Theorem 1. *Let* $k \in Z$, $f \in C^{|k|+2(m-1)-s}(D_R)$, $R > r$ *and* $f(z) = f_k(\rho)e^{ik\phi}$. *Assume that* f *satisfies equation (1) and* $f \equiv 0$ *in* D_r. *Then* $f \equiv 0$ *in* D_R.

By the next theorem we'll show that it is impossible to refine the conditions of Theorem 1.

Theorem 2. *For any* $\varepsilon \in (0, r)$ *there is* $f \in C^\infty(C)$ *with the next properties:*

1) the function f *satisfies (1) for* $z \in C$;

2) $f = 0$ *in* $D_{r-\varepsilon}$;

3) $f \not\equiv 0$.

The next result gives a description for all solutions (1) in a class $C^\infty(D_R)$ with one fixed $r < R$.

Theorem 3. *Let* $r > 0$, $m \in N$ *and* $s \in 0,...,m-1$ *are fixed. Let also* $R > r$ *and assume that a function* f *belongs to* $C^\infty(D_R)$.

Then the next statements are equivalent.

1) With $|z| < R - r$ *equality (1) holds.*

2) For any $k \in Z$ *on* $[0, R)$, *the next equality holds*

$$f_k(\rho) = \sum_{\substack{0 \leq p \leq s-1 \\ p+k \geq 0}} a_{k,p}\rho^{2p+k} + \sum_{p=0}^{m-s-1} b_{k,p}\rho^{2p+s+|k+s|} + \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k}\Phi_{\lambda,\eta,k}(\rho),$$

where $a_{k,p} \in C$, $b_{k,p} \in C$, $c_{\lambda,\eta,k} \in C$ *and*

$$c_{\lambda,\eta,k} = O(|\lambda|^{-\alpha})$$

as $\lambda \to \infty$ *for any fixed* $\alpha > 0$.

Note that analogues of the Theorem 3 for other equations related to ball mean values, were obtained by V.V.Volchkov for the first time (see [5–6] and the references in these papers).

Then let $Z(r_1, r_2) = Z_{r_1} \cap Z_{r_2}$.

We formulate now the local two-radii theorem for equation (1).

Theorem 4. *Let* $r_1, r_2 > 0$, $m \in N$ *and* $s \in 0,...,m-1$ *are fixed.*

Then:

1) *if* $R > r_1 + r_2$, $Z(r_1, r_2) = \varnothing$, $f \in C^{2m-2-s}(D_R)$ *and with* $|z| < R - r$ *holds (1), then* $f \in C^\infty(D_R)$ *and satisfies (2);*

2) *if* $\max\{r_1, r_2\} < R < r_1 + r_2$ *and* $Z(r_1, r_2) \neq \varnothing$, *then there is* $f \in C^\infty(D_R)$, *that satisfies (1) with* $|z| < R - r$ *and does not satisfy (2).*

For $m_1, m_2 \in N$, let

$Z_{r,m_i} = Z(g_{r,m_i}) \setminus \{z \in C : \arg z \in (-\pi, -\pi/2] \cup (\pi/2, \pi]\}$

where $i = 1, 2$.

Consider the case where $r_1 = r_2 = r$.

Theorem 5. *Let* $r > 0$, $m_1, m_2 \in N$, $m_1 \leq m_2$ *and* $s \in 0,...,m_1 - 1$.

Then:

1) *if* $R > 2r$, $Z_{r,m_1} \cap Z_{r,m_2} = \varnothing$, $f \in C^{2m_2-2-s}(D_R)$ *and for* $|z| < R - r$ *and fixed* $m = m_1, m_2$ *relation (1) is valid, then* $f \in C^\infty(D_R)$ *and (2) is satisfied for* $m = m_1$;

2) *if* $r < R < 2r$ *and* $Z_{r,m_1} \cap Z_{r,m_2} \neq \varnothing$, *then there is* $f \in C^\infty(D_R)$, *satisfying (1) with* $|z| < R - r$ *and fixed* $m = m_1, m_2$ *and does not satisfy (2) for* $m = m_1$.

3. Conclusions. The results in the present paper deal with the basic properties of functions satisfying relation (1) for certain values of parameters. In particular, we can assert that the only smooth function satisfying (1) and vanishing in D_r is zero. Moreover, the disk D_r in this statement cannot be changed by a smaller open disk with the same center.

The space of smooth solutions of (1) is described in Theorem 3. This result allows to obtain analogues of two-radii theorems for equation (1). In addition, a similar result for a single r holds, where two different values of m are considered.

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