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RESTRICTIONS ON QUIVERS OF EXPONENT MATRICES

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A concept of associated vertices of the quiver of an exponent matrix is introduced and the restrictions on total number of entering and outgoing arrows at such vertices are established. The restrictions on the maximum number of arrows of a quiver with the given number of loops are obtained, as well as on the number of arrows when each vertex of quiver must have a loop, and the restrictions on the number of simple oriented cycles of the quiver of an exponent matrix are obtained.

KEY WORDS: exponent matrix, quiver of matrix,

ОГРАНИЧЕНИЯ НА КОЛЧАНЫ ЭКСПОНЕНЦИАЛЬНЫХ МАТРИЦ

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В работе предложена концепция ассоциированных вершин колчанов экспоненциальных матриц и установлены ограничения на общее число входных и выходных стрел таких вершин. Получены ограничения на максимальное число стрел колчана с заданным числом циклов, на число стрел в случае, когда каждая вершина колчана должна иметь цикл, и на число простых ориентированных циклов в колчане экспоненциальной матрицы.

КЛЮЧЕВЫЕ СЛОВА: экспоненциальная матрица, колчан матрицы.

ОБМЕЖЕННЯ НА САГАЙДАКИ ЄКСПОНЕНЦІАЛЬНИХ МАТРИЦЬ

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В роботі запропонована концепція асоційованих вершин колчанів експоненціальних матриць і встановлені обмеження на загальне число вхідних і вихідних стріл таких вершин. Отримано обмеження на максимальне число стріл сагайдака із заданим числом циклів, на число стріл у випадку, коли кожна вершина сагайдака повинна мати цикл, і на число простих орієнтованих циклів в сагайдаку експоненціальної матриці.

КЛЮЧОВІ СЛОВА: експоненціальна матриця, сагайдак матриці.

1. Introducation. A notion of an exponent matrix is arisen from Ring theory. Exponent matrices are used in applied tasks, for example, by planning multifactor experiments under given optimum conditions, for a construction of codes et cetera. Each exponent matrix can be associated with some graph called a quiver and we can study such matrices by methods of Graph theory. A notion of a quiver was introduced by P. Gabriel in an investigation of finite-dimensional algebras with the square radical equals zero. A quiver of a reduced exponent matrix is simple laced (without multiple arrows and multiple loops) strongly connected oriented graph but not any such graph is a quiver of some exponent matrix [1]. In this paper we introduce a concept of associated vertices of a quiver and we establish restrictions on number of arrows, loops, and cycles of quivers of reduced exponent matrices.

The widespread use of methods of Computational Algebra is an important feature of this work. The named author makes computational programs and near two thousand examples of exponent matrices and their quivers are constructed with these programs. Hypotheses on properties of quivers of exponent matrices were based on an analysis of these examples. Obtained results were tested with these examples too.

Let $M_n(Z)$ be a ring of square $n \times n$ -matrices over the ring of integers.

A matrix $\varepsilon = (\alpha_{ij})$ from the ring $M_n(Z)$ is called *an exponent matrix* if the following conditions hold:

(i) $\alpha_{ii} = 0$ for all $i = 1, 2, \ldots, n$;

(ii) $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ for all $i, j, k = 1, 2, \ldots, n$.

Relations (ii) are called ring inequalities.

An exponent matrix $\varepsilon = (\alpha_{ij})$ is called *reduced* if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i \neq j$. Notice that it is enough to consider only reduced exponent matrices.

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Let $\varepsilon = (\alpha_{ij})$ be a reduced exponent matrix, E be the identity one. Denote $\varepsilon^{(1)} = \varepsilon + E = (\beta_{ij})$, $\varepsilon^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min_k \{\beta_{ik} + \beta_{kj} - \beta_{ij}\}$.

A graph Q is *a quiver* of a reduced exponent matrix ε if the adjacency matrix of Q is equal to $\varepsilon^{(2)} - \varepsilon^{(1)}$

Two exponent matrices are called *equivalent* if one of them can be obtain from other one under transformations of following two forms:

1) subtraction an integer from each element of some line and addition at the same time this integer to each element of the column with the same number;
2) permutation two lines and two columns with the same number simultaneously.

Note that each exponent matrix is equivalent to some exponent matrix with nonnegative elements. Note also that by first transformations a quiver of an exponent matrix does not change and by second transformations numbering of vertices changes but the form of a quiver does not change.

Note that one quiver can be associated with infinitely many exponent matrices.

By research on quivers, we consider a loop as an arrow from a vertex to the same vertex.

In this paper we establish restrictions on number of arrows, loops, and cycles of quivers of reduced exponent matrices.

Let $Q(\varepsilon)$ be a quiver of an exponent matrix $\varepsilon = (\alpha_{ij})$.

Now we introduce the following concept.

Definition. We will call vertices u and v $(u \neq v)$ of a quiver $Q(\varepsilon)$ *associated with an equality* $\alpha_{uv} + \alpha_{vu} = 1$ or simply *associated* if for elements α_{uv} and α_{vu} of the exponent matrix $\varepsilon = (\alpha_{ij})$ this equality holds true.

Will notice that a quiver $Q(\varepsilon)$ haven't loops at both associated vertices at the same time and each without loops vertex of quiver is associated with some other vertex and can be associated with a few different vertices.

For any vertex u of the quiver will denote by $od(u)$ and $id(u)$ number of arrows that go from the vertex u and, respectively, go to the vertex u.

Theorem 1. *Let* $Q(\varepsilon)$ *be a quiver of an exponent matrix* $\varepsilon = (\alpha_{ij})$ *and vertices* u *and* v *of* $Q(\varepsilon)$ *are associated. Then*

$$od(u) + od(v) \le n; \quad id(u) + id(v) \le n,$$

where n $(n \ge 2)$ *is the number of vertices of the quiver* $Q(\varepsilon)$.

Proof. The case $n = 2$ is trivial. Suppose $n \ge 3$; $\varepsilon = (\alpha_{ij})$; $[Q(\varepsilon)] = (q_{ij})$.

Since vertices u and v of the quiver $Q(\varepsilon)$ are associated, then the quiver $Q(\varepsilon)$ has no loops at vertices u and v and next equality holds true $\alpha_{uv} + \alpha_{vu} = 1$ that is either $\alpha_{uv} = 1$ and $\alpha_{vu} = 0$ or $\alpha_{vu} = 1$ and $\alpha_{uv} = 0$.

Let $\alpha_{uv} = 1$ and $\alpha_{vu} = 0$. We have by ring inequalities $\alpha_{vk} \le \alpha_{uk} \le \alpha_{vk} + 1$ for all $k \notin \{u, v\}$ and since elements of matrix $Q(\varepsilon)$ are integer, we have either $\alpha_{vk} = \alpha_{uk}$ or $\alpha_{uk} = \alpha_{vk} + 1$, from where it follows either $q_{uk} = 0$ or $q_{vk} = 0$ for all $k \notin \{u, v\}$. Consequently, $q_{uk} + q_{vk} \le 1$ for all $k \notin \{u, v\}$, from where, taking into account that $q_{uu} + q_{vv} = 0$, we have

$$\sum_{k=1}^{n} (q_{uk} + q_{vk}) = \sum_{k=1}^{n} q_{uk} + \sum_{k=1}^{n} q_{vk} = od(u) + od(v) \le n$$

.

Analogously

$$\sum_{k=1}^{n} (q_{ku} + q_{kv}) = id(u) + id(v) \le n.$$

In the case $\alpha_{vu} = 1$ and $\alpha_{uv} = 0$ a proof is analogous.

Theorem 1 is proved.

The next theorem gives the maximum number of arrows of a quiver with the given number of loops.

Theorem 2. *Suppose* $Q(\varepsilon)$ *is a quiver of an exponent matrix* ε, $Q(\varepsilon)$ *is a quiver on* n *vertices with* m *loops* $(n \ge 3; 0 \le m \le n-2)$, *and* K *denote the number of arrows (including loops) of the quiver* $Q(\varepsilon)$; *then*

$$K \le n^2 - 3n + m + 4. \tag{1}$$

Proof. The case $n = 2$ is trivial. Suppose $n \ge 3$. Will consider any two vertices of the quiver $Q(\varepsilon)$ that have no loops and are associated. Then the quiver $Q(\varepsilon)$ has necessarily $2n-2$ absent arrows of all possible arrows of a quiver. Taking into account absence $n-m-2$ more loops at the quiver $Q(\varepsilon)$, we obtain

$$K \le n^2 - (2n-2) - (n-m-2) = n^2 - 3n + m + 4$$

Theorem 2 is proved.

The next example shows that estimation (1) is exact for $n = 4$.

Example. For following exponent matrices ε_1, ε_2, ε_3 estimation (1) is an equality:

$$\varepsilon_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}; \quad [Q(\varepsilon_1)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix};$$

$$n = 4; \; m = 0; \; K = 8;$$

$$n^2 - 3n + m + 4 = 4^2 - 3 \cdot 4 + 0 + 4 = 8;$$

$$\varepsilon_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}; \ [Q(\varepsilon_2)] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix};$$

$$n = 4; \ m = 1; \ K = 9;$$

$$n^2 - 3n + m + 4 = 4^2 - 3 \cdot 4 + 1 + 4 = 9;$$

$$\varepsilon_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}; \ [Q(\varepsilon_3)] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix};$$

$$n = 4; \ m = 2; \ K = 10;$$

$$n^2 - 3n + m + 4 = 4^2 - 3 \cdot 4 + 2 + 4 = 10.$$

The next theorem gives a number of arrows when each vertex of quiver must have a loop.

Theorem 3. *Suppose* $Q(\varepsilon)$ *is a quiver of an exponent matrix* $\varepsilon = (\alpha_{ij})$, $Q(\varepsilon)$ *is a quiver on* n *vertices with* m *loops* $(n \ge 3; 0 \le m \le n)$, *and* K_1 *denote the number of arrows (without loops) of the quiver* $Q(\varepsilon)$. *If*

$$K_1 > n^2 - 3n + 4,$$

then each vertex of the quiver $Q(\varepsilon)$ *has a loop (that is* $m = n$ *and* $\alpha_{ij} + \alpha_{ji} > 1$ *for all* i, j*).*

Proof. It follows from Theorem 2 that the number of arrows (without loops) of the quiver $Q(\varepsilon)$ satisfies an inequality $K_1 \le n^2 - 3n + 4$ for $m \le n - 2$. Taking into account that a quiver on n vertices can't have $n - 1$ loops [1], for $K_1 > n^2 - 3n + 4$, we get the quiver $Q(\varepsilon)$ has a loop at each vertex (that is $m = n$ and $\alpha_{ij} + \alpha_{ji} > 1$ for all i, j [2]).

Theorem 3 is proved.

Theorem 4. *Will denote by* G *a simple oriented graph on* n *vertices with loops at* m *vertices* $(n \ge 3, 0 \le m \le n)$. *Suppose a graph* G *has arrows from each vertex to all other vertices; then the graph* G *is a quiver of an exponent matrix if and only if* $m = n$.

Proof. *Sufficiency.* Will consider a graph G with an adjacent matrix

$$[G] = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

It can easily be checked that this graph is a quiver of the exponent matrices

$$\varepsilon = \begin{pmatrix} 0 & k & \cdots & k \\ k & 0 & \dots & k \\ \dots & \dots & \ddots & \dots \\ k & k & \dots & 0 \end{pmatrix}, \ k \ge 1.$$

Necessity. Let the graph G be a quiver of an exponent matrix. Under the coditions of the theorem, the number K_1 of arrows (without loops) of the graph G is equal to $n^2 - n$. For $n \ge 3$, we have $K_1 = n^2 - n > n^2 - 3n + 4$ and it follows from Theorem 3 that such quiver has loop at each vertex.

Theorem 4 is proved.

The next theorem establishes restrictions on the number of simple oriented cycles of the quiver of an exponent matrix.

Theorem 5. *Suppose* $Q(\varepsilon)$ *is the quiver of an exponent matrix* ε, K_s *is the number of simple oriented cycles of the length* s *of the quiver* $Q(\varepsilon)$, K *is the number of all simple oriented cycles (including loops) of the quiver* $Q(\varepsilon)$*; then*

$$0 \le K_s \le P_{s-1} \cdot C_n^s = \frac{n!}{s \cdot (n-s)!};$$

$$1 \le K \le n! \cdot \sum_{s=1}^{n} \frac{1}{s \cdot (n-s)!},$$

where n *is the number of vertices of the quiver* $Q(\varepsilon)$.

Proof. Will consider exponent matrices of order n:

$$\varepsilon_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix},$$

$$\varepsilon_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

Quivers $Q(\varepsilon_1)$ and $Q(\varepsilon_2)$ of these matrices have adjacent matrices

$$[Q(\varepsilon_1)] = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$[Q(\varepsilon_2)] = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Note that the quiver of the exponent matrix ε_1 is a simple oriented cycle of the length n. Note also that the quiver $Q(\varepsilon_2)$ has only simple oriented cycles of the length 2 and it has not simple oriented cycles of other length.

It follows from the examples that the minimal possible number of simple oriented cycles of the length s of a quiver of a reduced exponent matrix equals zero.

The maximum possible number of arrows of a quiver on n vertices is equal to n^2. Will consider exponent matrix of order n:

$$\varepsilon_3 = \begin{pmatrix} 0 & k & k & \dots & k \\ k & 0 & k & \dots & k \\ k & k & 0 & \dots & k \\ \dots & \dots & \dots & \ddots & \dots \\ k & k & \dots & k & 0 \end{pmatrix}, \quad k \geq 1 .$$

Quiver $Q(\varepsilon_3)$ of this matrix has adjacent matrix

$$[Q(\varepsilon_3)] = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

that is for any n, there are quivers on n vertices with n^2 arrows.

The number of ways, in which s different vertices being out of order can be selected of n vertices, is equal to the number of combinations of n things s at a time:

$$C_n^s = \frac{n!}{s!(n-s)!} .$$

The number of different simple oriented cycles on s selected vertices is equal to the number of permutations of $s-1$ elements:

$$P_{s-1} = (s-1)! .$$

Consequently, the maximum possible number of simple oriented cycles of the length s of a quiver $Q(\varepsilon)$ equals

$$P_{s-1} \cdot C_n^s = (s-1)! \cdot \frac{n!}{s! \cdot (n-s)!} = \frac{n!}{s \cdot (n-s)!}$$

and

$$0 \leq K_s \leq P_{s-1} \cdot C_n^s = \frac{n!}{s \cdot (n-s)!} .$$

Since a quiver of a reduced exponent matrix is always strongly connected, it always has at least one oriented cycle. The maximum possible number of simple oriented cycles of a quiver $Q(\varepsilon)$ equals

$$\sum_{s=1}^{n} \frac{n!}{s \cdot (n-s)!} = n! \cdot \sum_{s=1}^{n} \frac{1}{s \cdot (n-s)!} .$$

Thus

$$1 \leq K \leq n! \cdot \sum_{s=1}^{n} \frac{1}{s \cdot (n-s)!} .$$

Theorem 5 is proved.

The obtained results are theoretical, and they can be used for an investigation in Representation Theory and in modern Structural Theory of Rings. In the future we plan to carry out further investigation of the formulated problem.

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