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(0,1)−ORDERS AND PARTIALLY ORDERED SETS

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A structure of (0,1)-orders and their quivers using the partially ordered sets has been studied. The form of the Pierce decomposition of a reduced (0,1)-order has been established. A construction that makes it possible to construct a strongly connected quiver without multiple arrows from the diagram of a finite partially ordered set has been introduced and it was proved the obtained quiver coincides with the quiver of the corresponding (0,1)-order.

KEY WORDS: quver, partially ordered set, (0,10-orders.

(0,1)−ПОРЯДКИ И ЧАСТИЧНО УПОРЯДОЧЕННЫЕ МНОЖЕСТВА

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В работе приведены результаты исследования структуры (0,1)-порядков и их колчанов с помощью частично ориентированных множеств. Предложен подход, позволяющий конструировать строго связанные колчаны без множественных стрел из диаграмм конченых частично ориентированных множеств. Доказано, что полученные колчаны совпадают с колчаном соответствующего (0,1)-порядка.

КЛЮЧЕВЫЕ СЛОВА: колчан, частично упорядоченное множество, (0,1)-порядки.

(0,1)−ПОРЯДКИ І ЧАСТКОВО ВПОРЯДКОВАНІ МНОЖИНИ

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В роботі наведені результати дослідження структури (0,1)-порядків і їх сагайдаків за допомогою частково орієнтованих множин. Запропоновано підхід, який дозволяє конструювати строго пов'язані сагайдаки без множинних стріл з діаграм кінчених частково орієнтованих множин. Доведено, що отримані сагайдаки збігаються зі сагайдаком відповідного (0,1) -порядка.

КЛЮЧОВІ СЛОВА: сагайдакі, частково орієнтована множина, (0,1)-порядки.

1. Introduction. Recently Representation Theory of finite-dimensional Algebras is quickly developed. Powerful constructive methods of this Theory are used more in other areas of Mathematics, in particular, in Representation Theory of various Algebraic Structures, in Ring Theory. In this paper we research a structure of (0,1)-orders and their quivers with the help of partially ordered sets. We use methods of Quiver Theory and Ring and Module Theory. The use of methods of Computational Algebra is an important feature of this work.

2. Main results. Let $\Lambda = \{\vartheta, \varepsilon(\Lambda)\}$ be a tiled order over a discrete valuation ring ϑ with an exponent matrix $\varepsilon(\Lambda) = (\alpha_{ij})$, where α_{ij} are integers and they satisfy relations $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ for all i, j, k (these relations are called ring inequalities), $\alpha_{ii} = 0$ for all i [1]. For convenience will say "order" considering it as tiled order.

Note that by studying of tiled orders, it is enough to consider only reduced orders, for exponent matrices of which it is $\alpha_{ij} + \alpha_{ji} > 0$ for all $i \neq j$. That is why we shall assume that an order Λ is reduced that is the factor-ring A / R is a direct sum of skew fields.

Two exponent matrices are called *equivalent* if one of them can be obtain from other one under transformations of following two forms: subtraction an integer from each element of some line and addition at the same time this integer to each element of the column with the same number; permutation two lines and two columns with the same number simultaneously.

Note that any exponent matrix is equivalent to some exponent matrix with nonnegative elements.

A tiled order $\Lambda = \{\vartheta, \varepsilon(\Lambda)\}$ is called (0,1)-order if the exponent matrix $\varepsilon(\Lambda)$ is a (0,1)-matrix.

With such (0,1)-order Λ we associate a partially ordered set $S(\Lambda)$ of n elements $S(\Lambda) = \{a_1, a_2, \ldots a_n\}$ with an ordering relation $\leq$ defined by the following way: $a_i \leq a_j \Leftrightarrow \alpha_{ij} = 0$.

Conversely, if there are a finite partially ordered set $S=\{a_1,a_2,\dots a_n\}$ with an ordering relation $\le$ and a discrete valuation ring ϑ, then we can construct (0,1)-order $\Lambda=\{\vartheta,\varepsilon(\Lambda)\}$ by the rule:

$$\alpha_{ij}=\begin{cases}0, & \text{if } a_i\le a_j,\\ 1 & \text{otherwise}.\end{cases}$$

Give the definition of a width of a partially ordered set. Consider the maximum possible number of elements of a subset of the set S that consist of pairwise noncomparable elements. If this number is finite, then it is called *the width* of the partially ordered set S and is denoted by $w(S)$.

Theorem 1. *Let Λ be a reduced (0,1)-order. Then up to an equivalence, the exponent matrix of Λ has the form:*

$$\varepsilon(\Lambda)=\begin{pmatrix}H_{n_1} & R_{12} & \cdots & R_{1t}\\ R_{21} & H_{n_2} & \cdots & R_{2t}\\ \dots & \dots & \dots & \dots\\ R_{t1} & R_{t2} & \cdots & H_{n_t}\end{pmatrix},\quad \textit{where}$$

$$H_{n_i}=\underbrace{\begin{pmatrix}0 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\\ \dots & \dots & \ddots & \dots\\ 1 & 1 & \dots & 0\end{pmatrix}}_{n_i},$$

$t=w(S(\Lambda))$ *is the width of the corresponding partially ordered set* $S(\Lambda)$, R_{kl}, $k,l=\overline{1,t}$, $k\ne l$, *is a (0,1)-matrix of order* $n_k\times n_l$ *such that any element* β_{ij} *of a matrix* R_{kl} *and the element* γ_{ji} *of the matrix* R_{lk} *satisfy the condition* $\beta_{ij}+\gamma_{ji}>0$.

By the proof of Theorem 1, we use the Dilworth theorem [2].

Theorem 2 (Dilworth). *The minimal number of nonintersecting paths, which contain all elements of partially ordered set* S *in total, is equal to maximum possible number of elements of a subset of the set* S *that consist of pairwise noncomparable elements if this number is finite.*

Proof of Theorem 1. Let t be the width of the partially ordered set $S(\Lambda)$ associated with a (0,1)-order Λ. Let n_1, n_2,..., n_t be the numbers of elements of firstly, secondly,…, t-ly path respectively ($n_1+n_2+\ldots+n_t=n$). Enumerate elements of $S(\Lambda)$ by the following way: at first enumerate elements of the first path from 1 to n_1, next enumerate elements of the second path from n_1+1 to n_1+n_2, and so on.

Then (0,1)-order Λ has such exponent matrix:

$$\varepsilon(\Lambda)=\begin{pmatrix}H_{n_1} & R_{12} & \dots & R_{1t}\\ R_{21} & H_{n_2} & \dots & R_{2t}\\ \dots & \dots & \ddots & \dots\\ R_{t1} & R_{t2} & \dots & H_{n_t}\end{pmatrix}.$$

Since the matrices H_{n_i} correspond to paths, they have the form

$$H_{n_i}=\underbrace{\begin{pmatrix}0 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\\ \dots & \dots & \ddots & \dots\\ 1 & 1 & \dots & 0\end{pmatrix}}_{n_i}.$$

Under the condition of Theorem 1, order Λ is reduced that is the matrix $\varepsilon(\Lambda)$ has no symmetric zeros. Because of this, for any element β_{ij} of a matrix R_{kl} and the corresponding element γ_{ji} of the matrix R_{lk}, the inequality $\beta_{ij}+\gamma_{ji}>0$ holds.

Theorem 1 is proved.

Give the definition of a Gabliel quiver of a ring.

Suppose A is a semiperfect right Noetherian ring; R is its Jacobson radical, P_1, P_2, … , P_n are all pairwise nonisomorphic indecomposable projective modules. Suppose a projective cover $P(P_iR)$ of the module P_iR is

$$P(P_iR)=\bigoplus_{j=1}^{n}P_j^{t_{ij}},\quad i,j=1,2,\dots,n.$$

We put points (vertices) $1,2,\dots,n$ in correspondence to the modules P_1, P_2, … , P_n, and join a vertex i with a vertex j by t_{ij} arrows.

The constructed graph is called *a quiver* of a semiperfect right Noetherian ring A and is denoted by $Q(A)$.

Let $\Lambda=\{\vartheta,\varepsilon(\Lambda)\}$ be a reduced tiled order with the quiver $Q(\Lambda)$. It is known that such quiver is a strongly connected oriented graph without multiple arrows and adjacency matrix of the quiver $Q(\Lambda)$ is equals to the difference of exponent matrices of the squared Jacobson radical and of the Jacobson radical of the order Λ:

$$[Q(\Lambda)]=\varepsilon(R^2)-\varepsilon(R).$$

Recall that a quiver is called strongly connected if there is a path between two its arbitrary vertices. In addition, one vertex without loops is considered also as a strongly connected quiver.

Give the definition of a diagram of a finite partially ordered set.

Let $S=\{a_1,a_2,\dots a_n\}$ be a finite partially ordered set with an ordering relation $\le$. A quiver $Q(S)$ with a set of vertices $VQ(S)=\{1,2,\dots n\}$ and with a set of arrows $AQ(S)=\{\sigma\}$ is called *the diagram of* S if the following condition hold: an arrow $\sigma:i\to j$ ($i\ne j$) belong to $AQ(S)$ if and only if $a_i\le a_j$ and there is no element $a_k\in S$ such that $a_i\le a_k\le a_j$, where $a_k\ne a_i$, $a_k\ne a_j$.

Note that a quiver $Q(S)$ has no oriented cycles in particular loops that is $Q(S)$ is an acyclic quiver without multiple arrows. An arrow $\sigma: i \to j$ of an acyclic quiver Q is called *unnecessary* if there is a path from the vertex i to the vertex j of the length greater than 1.

Statement. Let Q be an acyclic quiver without multiple and unnecessary arrows; then Q is the diagram of a finite partially ordered set. Conversely, a diagram Q(S) of a finite partially ordered set S is an acyclic quiver without multiple and unnecessary arrows.

Now we introduce a construction that makes it possible to construct a strongly connected quiver without multiple arrows from the diagram Q(S) of a finite partially ordered set $S = \{a_1, a_2, \ldots a_n\}$.

Will denote by S_{max} the set of all maximal elements of a partially ordered set S, by S_{min} the set of all minimal elements of S, by $S_{max} \times S_{min}$ the Cartesian product of sets S_{max} and S_{min}, and by $\tilde{Q}(S)$ a quiver obtained from the diagram Q(S) by adding all arrows $\sigma_{ij}: a_i \to a_j$ for all $(a_i, a_j) \in S_{max} \times S_{min}$.

It is clear that the quiver $\tilde{Q}(S)$ is a strongly connected quiver without multiple arrows.

A tiled (0,1)-order corresponding to a partially ordered set S will be denoted by $\Lambda(S)$.

Theorem 3. *A quiver* $Q(\Lambda(S))$ *coincides with the quiver* $\tilde{Q}(S)$.

Proof. We recall that $[Q(\Lambda)] = \varepsilon(R^2) - \varepsilon(R)$.

Suppose the diagram Q(S) has an arrow from a vertex s to a vertex t. This means that $\alpha_{st} = 0$ and there is no number k ($k \neq s, t$) such that $\alpha_{sk} = 0$ and $\alpha_{kt} = 0$. The elements β_{ss} and β_{tt} of the exponent matrix $\varepsilon(R) = (\beta_{ij})$ are equal to 1. Let $\varepsilon(R^2) = (\gamma_{ij})$. As $\beta_{ij} = \alpha_{ij}$ for $i \neq j$, we have $\gamma_{st} = \min\limits_{1 \le k \le n} \{\beta_{sk} + \beta_{kt}\} = 1$. Then $\beta_{st} = \alpha_{st} = 0$ and $\gamma_{st} = 1$. Therefore the quiver $Q(\Lambda(S))$ has an arrow from the vertex s to the vertex t.

Let $a_p \in S_{max}$. Then $\alpha_{pk} = 0$ only in the case $k = p$. That is why p-th row of the matrix $\varepsilon(R)$ consist of unities that is $(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pn}) = (1, \ldots, 1, \ldots, 1)$. Analogously, if $a_q \in S_{min}$, then q-й column $(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T$ of the matrix $\varepsilon(R)$ equals $(1, \ldots, 1, \ldots, 1)^T$. Then we have $\gamma_{pq} = 2$ and therefore the quiver $Q(\Lambda(S))$ has an arrow from the vertex p to the vertex q.

Thus we prove the inclusion $\tilde{Q}(S) \subseteq Q(\Lambda(S))$.

Will prove the converse inclusion. Let $\gamma_{pq} = 2$. Clearly, then $(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pn}) = (1, \ldots, 1, \ldots, 1)$ and $(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T = (1, \ldots, 1, \ldots, 1)^T$, from where it follows that $a_p \in S_{max}$, $a_q \in S_{min}$, and there is an arrow from the vertex p to the vertex q.

Let $\gamma_{pq} = 1$ and $\beta_{pq} = 0$. As $\gamma_{pq} = \min\limits_{1 \le k \le n} \{\beta_{pk} + \beta_{kq}\} = 1$, then $\beta_{pk} + \beta_{kq} \ge 1$ for all $k = 1, 2, \ldots, n$. For $k = p$ and $k = q$, we have $\beta_{pp} + \beta_{pq} = \beta_{pq} + \beta_{qq} = 1$. Therefore we can consider $k \neq p, q$. Thus $\beta_{pq} = \alpha_{pq} = 0$ that is $a_p \le a_q$ and the inequality $\alpha_{pk} + \alpha_{kq} \ge 1$ means that the diagram Q(S) has an arrow from the vertex p to the vertex q.

This prove the inclusion $Q(\Lambda(S)) \subseteq \tilde{Q}(S)$.

Thus $Q(\Lambda(S)) = \tilde{Q}(S)$.

Theorem 3 is proved.

It is follows from the introduced above method of a construction of a quiver of (0,1)-order from the diagram of the corresponding partially ordered set and from Theorem 3 the next statements.

3. Statements:

1. The number of loops of a quiver of (0,1)-order is equal to the number of elements of the corresponding partially ordered set that are minimal and maximal simultaneously or it is equal to the number of isolated vertices of the diagram of the corresponding partially ordered set.
2. A quiver of (0,1)-order has no loops if and only if the corresponding partially ordered set has no elements that are minimal and maximal simultaneously or the diagram of the corresponding partially ordered set has no isolated vertices.
3. For any positive integer m ($m \ge 2$), there is no (0,1)-order with a quiver on m vertices with $m-1$ loops.

4. Conclusions. The obtained results are theoretical and they can be used for the development of methods of Quiver Theory in the modern structural Ring Theory. These results can be also used by giving specialized courses in algebra.

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