DIVISORS OF THE GAUSSIAN INTEGERS IN NORM GROUP E_n^+

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Let A, B be two infinite sets of positive numbers. We define generalized function of divisors

$$\tau_{\mathrm{A,B}}(n) = \#\{(a,b)\in \mathrm{A}\times\mathrm{B} \mid ab=n\}, \ (n\in\mathbb{N}).$$

In the case $\mathrm{A}=\mathrm{B}=\mathrm{N}$ we have the classical Dirichlet problem of divisors. In works of Smith and Subbarao [2], Nowak [1], Varbanec and Zarzycki [3] investigated the case $\mathrm{A}=\mathrm{N}$, $\mathrm{B}=\mathrm{B}(b_0,q) := \{b\in\mathbb{N} \mid b\equiv b_0 \pmod q\}$. In the sequel came to be consider other sets A and B.

Varbanec and Zarzycki [4], Varbanec [5] generalized this problem on the case of sets A, B, which define as the sets of all positive integers each of which is norm of integer ideal in finite extension of field $\mathbb{Q}$.

We will consider a generalized function of divisors over the ring of Gaussian integers determined in this way: for every $w\in\mathbb{Z}[i]$ we put

$$\tau(w;E_n^+) = \sum_{\substack{\delta\mid w\\ \delta\in E_n^+}} 1,$$

where

$$E_n^+ := \left\{\alpha\in\mathbb{Z}[i] \,\middle|\, \mathrm{N}(\alpha)\equiv 1\left(\mathrm{mod}\, p^n\right)\right\},$$

($p\equiv 3(\mathrm{mod}\,4)$, p is prime).

The set E_n^+ is a multiplicative subgroup in the multiplicative group of classes of residues modulo p^n over $\mathbb{Z}[i]$.

We denote

$$\boldsymbol{\tau}^{(m)}\left(w;E_n^+\right) := \sum_{\substack{\boldsymbol{\alpha}\in E_n^+\\ \boldsymbol{\alpha}\mid w}} e^{4mi\arg w} = e^{4mi\arg w}\boldsymbol{\tau}^{(0)}\left(w;E_n^+\right) = e^{4mi\arg w}\boldsymbol{\tau}\left(w;E_n^+\right).$$

Theorem 1. Let p be a prime rational number, $p\equiv 3(\mathrm{mod}\,4)$. For any positive integer n the asymptotic formula

$$\sum_{N(w)\le x}\tau^{(m)}\left(w;E_n^+\right) = \varepsilon_m\left(\frac{\pi^2}{2}\frac{p+1}{p}\frac{x\log x}{p^n} + \frac{\pi x}{4p^n}\frac{p+1}{p}\left(b_0(\tilde{\chi}_0)+\gamma+L'(1,\chi_4)\right)+O\left(x^{1/2+\varepsilon}\log T\right) + O\left(\left(\frac{T^2+m^2}{T^2}\right)^{\frac12}p^{-n}x^{\frac12}\right)\right)$$

holds. Where

$$\varepsilon_m = \begin{cases} 1 \ if \ m=0\\ 0 \text{ else}\end{cases},$$

a parameter $b_0(w) = \begin{cases} 1 \ if \ w\in E_n^+\\ 0 \text{ else}\end{cases}$

Theorem 2. For $p^n \ll x^{\frac12-\varepsilon}$ and $(\varphi_2-\varphi_1) \ll \left(xp^{-2n}\right)^{-\frac12+\varepsilon}$ the following asymptotic formula

$$\sum_{\substack{\alpha\in \mathrm{E}_n^+\\ \phi_1\le\arg w\le\phi_2\\ \mathrm{N}(w)\le x}} \tau\left(\mathrm{w};\alpha,\mathrm{E}_n^+\right) = \left(\frac{\pi^2 \mathrm{x}\log \mathrm{x}}{\mathrm{p}^n}\cdot\frac{\mathrm{p}+1}{\mathrm{p}} + \frac{\mathrm{x}}{\mathrm{p}^{2n}}\sum_{\alpha\in\mathrm{E}_n}\mathrm{c}\left(\alpha,\mathrm{p}^n\right)\right)\times (\phi_2-\phi_1) + \mathrm{O}\left(\mathrm{x}^{\frac12+\varepsilon}\right)$$

The constant in symbol “O” depends only ε, $\varepsilon>0$.

References

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