L-INDEX IN DIRECTION OF FUNCTION OF FORM $f(\sqrt{z_1 z_2})$

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Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a given direction, $L(z): \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function, $F: \mathbb{C}^n \to \mathbb{C}$ be an entire function, $g_{z^0}(t) := F(z^0 + t\mathbf{b})$, $l_{z^0}(t) := L(z^0 + t\mathbf{b})$, $t \in \mathbb{C}$.

Definition (see [1]). An entire function $F(z)$, $z \in \mathbb{C}^n$, is said to be of *bounded* L*-index in a direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ next inequality is true:

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \quad (1)$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\right), k \ge 2.$ The least such integer m_0 is called a L-index in direction $\mathbf{b}$ of $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L)$. If m_0 does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$ and F is said of unbounded L-index in direction. $\mathbf{b}$. If $n = 1$, $\mathbf{b} = 1$ and $L(z) = l(z)$, $z \in \mathbb{C}$, an inequality (1) defines a bounded l-index with l-index $N(F, l) \equiv N_1(F, l)$ [2]. And in the case $L(z) \equiv 1$ we get a notion of bounded index with index $N(F) \equiv N_1(F, 1)$.

Exploring properties of entire functions of bounded L-index in direction, we obtained the following assertion.

Theorem 1. [1] *An entire function* $F(z)$ *is of bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exists a number* $M > 0$ *that for all* $z^0 \in \mathbb{C}^n$ *the function* $g_{z^0}(t)$ *is of bounded* l_{z^0}*-index with* $N(g_{z^0}, l_{z^0}) \le M < +\infty$ *as a function of variable* $t \in \mathbb{C}$ *and* $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{C}^n\}$.

In view of Theorem 1, there was a natural *question* [3]: *is there an entire function* $F(z)$, $z \in \mathbb{C}^n$, *with* $N(g_{z^0}, l_{z^0}) < +\infty$ *for every* $z^0 \in \mathbb{C}^n$, *but* $N_{\mathbf{b}}(F, L) = +\infty$ *?*

We gave an affirmative answer [3] to the above-mentioned question. There was proved that $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ is of unbounded index in the direction $(1,1)$. Recently, this result was generalized for each direction $\mathbf{b} \in \mathbb{C}^2 \setminus \{0\}$ [4].

At a scientific seminar in Lviv University prof. Kondratyuk A. A. asked the question (2007):

What is a positive continuous function $L: \mathbb{C}^2 \to \mathbb{R}_+$ *providing a bounded* L*-index in the direction* $\mathbf{b}$ *of function* $\cos\sqrt{z_1 z_2}$?

Using Skaskiv's idea from [3], we give an answer to Kondratyuk's question in the following theorem.

Theorem 2. *Let* $\varepsilon > 0$,

$$L_\varepsilon(z_1, z_2) = \begin{cases} |b_1\sqrt{\frac{z_2}{z_1}} + b_2\sqrt{\frac{z_1}{z_2}}| + 1, & |z_1 z_2| > \varepsilon^2, \\ \frac{|b_1 z_2 + b_2 z_1|}{\varepsilon} + 1, & |z_1 z_2| \le \varepsilon^2. \end{cases}$$

Then $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ *is of bounded* L_ε*-index in the direction* $\mathbf{b} = (b_1, b_2)$.

In view of Theorem 1 and 2 the following hypothesis arises

Theorem. *Let* $\varepsilon > 0$, $f(t)$ *be an entire function of bounded index* $(t \in \mathbb{C})$,

$$L_\varepsilon(z_1, z_2) = \begin{cases} |b_1\sqrt{\frac{z_2}{z_1}} + b_2\sqrt{\frac{z_1}{z_2}}| + 1, & |z_1 z_2| > \varepsilon^2, \\ \frac{|b_1\sqrt{\frac{z_2}{z_1}} + b_2\sqrt{\frac{z_1}{z_2}}| \cdot |f'(\sqrt{z_1 z_2})|}{f'(\varepsilon e^{\frac{i}{2} arg(z_1 z_2)})} + 1, & |z_1 z_2| \le \varepsilon^2. \end{cases}$$

Then $F(z_1, z_2) = f(\sqrt{z_1 z_2})$ *is of bounded* L_ε*-index in the direction* $\mathbf{b} = (b_1, b_2)$.

LITERATURE

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