Generalization and application of Wiman-Valiron's method for fractional derivatives of entire functions

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Motivation

- Fractional differential equations are widely used for modeling anomalous relaxation and diffusion phenomena; see [3] for further references.
- The mathematical theory of such equations is still in its initial stage. A systematic development of the analytic theory of fractional differential equations with variable coefficients was initiated only recently, in the paper by Kilbas, Rivero, Rodríguez-Germá, and Trujillo [1] (see also Section 7.5 in [2]).

Riemann-Liouville operator

Let $f \in L(0,a)$, $a > 0$. The *Riemann-Liouville fractional derivative* of order $\alpha > 0$ for f is defined as

$$D^{\alpha} f(x) = \frac{d^n}{dx^n}\{I^{n-\alpha} f(x)\}, \quad \alpha \in (n-1, n], \quad n \in \mathbb{N},$$

where

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}$$

is the *Riemann-Liouville fractional integral* of order $\alpha > 0$ for h, $\Gamma(\alpha)$ is the Gamma function. In particular, if $0 < \alpha < 1$, then

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x \frac{f(t)dt}{(x-t)^{\alpha}}.$$

Fractional derivatives of an entire function

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\theta} \qquad (1)$$

be an entire function. The fractional derivative has the following property ([4])

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad (2)$$

where $\alpha > 0, \beta > -1$.
From (2) the fractional derivative for the entire function (1) w.r.t. $r = |z|$ is defined as

$$|z|^{\alpha} D^{\alpha} f(z) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} z^n.$$

Remark. $D^{\alpha} c$ has a singularity at the origin.

Wiman-Valiron settings

For $r \in [0, +\infty)$ we denote
the maximum modulus $M(r, f) = \max\{|f(z)| : |z| = r\}$,
the maximal term $\mu(r, f) = \max\{|a_n| r^n : n \geq 0\}$,
the central index $\nu(r, f) = \max\{n \geq 0 : |a_n| r^n = \mu(r, f)\}$ of the series (1).
Let $(\alpha_n)_{n=0}^{\infty}$ be a sequence of positive numbers such that α_{n+1}/α_n decreases with increasing n. Let (ϱ_n) be a sequence of numbers such that

$$0 < \varrho_0 < \frac{\alpha_0}{\alpha_1}, \quad \frac{\alpha_{n-1}}{\alpha_n} < \varrho_n < \frac{\alpha_n}{\alpha_{n+1}}, \quad (n \geq 1)$$

so that (ϱ_n) increases with increasing n. We shall say that a value r is normal (for the sequence (a_n),(α_n) and (ϱ_n)) if we have for some ν

$$|a_n| r^n \leq |a_\nu| r^\nu \frac{\alpha_n \varrho_\nu^n}{\alpha_\nu \varrho_\nu^\nu} \quad (n \geq 0).$$

Wiman-Valiron settings

Let V be the class of positive continuous nondecreasing functions v on $[0, +\infty)$ such that $\frac{x^2}{v(x)\ln v(x)}$ increases to $+\infty$ on $x \in [x_0; +\infty)$, $x_0 > 0$, and $\int\limits_0^{+\infty} \frac{dx}{v(x)} < +\infty$. For example, the function $v(x) = x\ln^{\alpha+1} x$, $(x \geq e)$, $\alpha \in (0, 1)$ belongs to V .

Main result

Theorem 1. Let $v \in V$ and $\varkappa(t) = 4\sqrt{v(t)\ln v(t)}$. Suppose that f is entire, a value r is normal and enough large, $|z_0| = r$,

$$|f(z_0)| \geq \eta M(r,f), \quad v^{-2}(\nu(r,f)) \leq \eta \leq 1$$

holds, and

$$r\left(1 - \frac{1}{40\varkappa(\nu)}\right) < \varrho < r\left(1 + \frac{1}{40\varkappa(\nu)}\right), \quad \nu = \nu(r,f).$$

Then if $q > 0$ we have for $|z| = \varrho$

$$\left(\frac{\rho}{\nu}\right)^q D^q f(z) = f(z) + O\left(\frac{\varkappa(\nu)}{\nu}\right) M(\varrho, f). \qquad (3)$$

In particular, if $\ln\varrho - \ln r = o\left(\frac{1}{\varkappa(\nu)}\right)$ then

$$M(\varrho, |z|^q D^q f(z)) = \nu^q \left\{1 + O\left(\frac{\varkappa(\nu)}{\nu}\right)\right\} M(\varrho, f) \sim \nu^q M(r,f)$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

Proof ingredients

We follow Hayman's exposition [5] with modifacations from [6]. Let

$$\nu_1 = \min\{n : |n-\nu| \le \varkappa(\nu)\}, \quad \nu_2 = \max\{n : |n-\nu| \le \varkappa(\nu)\}.$$

We write

$$P(z) = \sum_{|n-\nu|\le\varkappa(\nu)} |a_n| z^{n-\nu_1},$$

$$f(z) = P(z)z^{\nu_1} + R(z). \tag{4}$$

Estimating the first summand in (4), the key role plays generalized Leibnitz's formula for fractional derivatives ([4], p.216). Let $f(x)$ and $g(x)$ be analytic functions on $[a, b]$, then

$$D^\alpha(f\cdot g) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} (D^{\alpha-k}f)g^{(k)}, \tag{5}$$

where $\binom{\alpha}{k} = \dfrac{(-1)^k \alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}$.

We consider the fractional differential equation in the form

$$\frac{\widetilde{D}^q(r^q f(z))}{z} + a(z)f(z) = 0, \tag{6}$$

where the coefficient $a(z)$ is an entire function, $q > 0$, and

$$\widetilde{D}^q f(z) = D^q f(z) - \Gamma(q+1)f(0). \tag{7}$$

The proofs of the following theorems are standard ([7]).
Theorem 2. The equation (6) with the initial condition $f(0) = f_0$ has an entire solution.
Theorem 3. Let $a(z)$ be a polynomial of degree $m \geq 0$. Then all not-trivial solutions f of the equation (6) have the order of growth $\varrho = \frac{m+1}{q}$.

References

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- Thank you for your attention!