STABILITY OF WAVE-PACKETS IN THE TWO-LAYER FLUID WITH FREE SURFACE AND RIGID BOTTOM

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Identification of the conditions of the stability is of considerable theoretical and practical interest of researchers of gravitational waves in the World Ocean as well as capillary waves in applied investigations.

In the article [1] the two-layer system was considered on the basis of the Euler equations when the thickness is small is considered the horizontal scale, the solutions were founded as series respected to the small thickness parameter. Nonlinear internal waves on the surface of contact of two semi-infinite fluids with different densities were investigated in [2], the solution based on the Fourier series expansion of the unknown functions; the main characteristics of wave motions in some limiting cases were investigated. Propagation of internal waves in the two-layer fluid bounded from above and below the solid lids was investigated in the article [3] without the surface tension. The solution was obtained in the form of generalized power series of the parameter, which depends on the value inversed to the Froude number.

A.Nayfeh [4] used the method of multiple scales to obtain a pair of partial differential equations that describe the evolution of finite-amplitude wave-packets on the interface of two semi-infinite fluids with different densities taking into account the effect of surface tension. As a result two alternative nonlinear Schrödinger equations were obtained and stability of finite amplitude wave-packets was investigated. H.Hasimoto & H.Ono have used the method of multiple scales to obtain the nonlinear Schrödinger equation, which describes the evolution of gravity wave packets of finite amplitude on the surface of fluid layer. The essential contribution to the study of this problem was made by the H.Segur & D.Hammakc [6], H.Yuen & B.Lake [7], M.Ablowitz & H.Segur [8], J.Whitham [9], P.Bhatnagar [10], D.Lamb [11], I.T.Selezov & S.V.Korsunsky [12], [13]. Propagation of wave packets in a fluid environment, taking into account surface tension was described in articles [14], [15].

Accounting surface tension plays an important role at investigation of the capillary waves at two-component hydrodynamic systems in laboratory researches. Also, the works [16–21] are devoted to the problem of propagation nonlinear internal waves.

The investigations of wave packet propagation on the free surface and the contact surface of two-fluid systems were presented at [4, 22–30]. In the article [4] the analysis of propagation of wave-packets on the interface of two semi-infinite fluids with different densities is carried out taking into account the effect of surface tension. A similar problem of propagation of wave-packets on the interface of half-space and the layer was studied in [22], where the problem of stability of wave packets in the system "layer – a half-space" using the method of multiple scales up to the third order was studied; the case of small frequencies was investigated in [23].

In the papers published recently various aspects of the fourth approximation of the problems of evolution of nonlinear wave-packets were considered, such as, evolution equation near the cut-off critical wave number was obtained in [24], [25], evolution equation for wave numbers far from the critical was obtained in [26], [27], investigating the stability of the solutions of these equations in the system "layer – a half-space" was carried out [28]. Investigation of stability for the system of "layer – layer" performed in the works [29], [30].

In this paper the problem of nonlinear stability in the system "layer with rigid bottom – layer with free surface" is considered.

1. Statement of the problem and method of solution. The mathematical statement of the problem for wave-packet propagation along the interface between the upper layer with free surface and a lower layer with free surface is presented as nonlinear boundary value problem

$$\nabla^2\phi_j = 0 \ \text{ in } \ \Omega_j ,$$
$$\eta_{,t} - \phi_{j,z} = -\alpha\phi_{j,x}\eta_{,x} \quad \text{at } z = \alpha\eta(x,t) ,$$
$$\eta_{0,t} - \phi_{2,z} = -\alpha\phi_{2,x}\eta_{0,x} \quad \text{at } z = \alpha\eta_0(x,t) ,$$
$$\phi_{1,t} - \rho\phi_{2,t} + (1-\rho)\eta + 0.5\alpha\left[(\nabla\phi_1)^2 - \rho(\nabla\phi_2)^2\right] -$$
$$-T\left(1+\alpha^2\eta_{,x}^2\right)^{-3/2}\eta_{,xx} = 0 \quad \text{at } z = \alpha\eta(x,t) , \tag{1}$$
$$\phi_{2,t} + \eta_0 + 0.5\alpha(\nabla\phi_2)^2 - T_0\left(1+\alpha^2\eta_{0,x}^2\right)^{-3/2}\eta_{0,xx} = 0 \quad \text{at } z = \alpha\eta_0(x,t) ,$$
$$\phi_{1,z} = 0 \ \text{ at } \ z = -h_1 ,$$

where ϕ_j ($j = 1, 2$) are the velocity potentials; η and η_0 are the elevations of the interface and the free surface; $\rho = \rho_2 / \rho_1$ is ratio of fluids densities; $\alpha = a / L$ is the nonlinearity coefficient; the lower fluid layer $\Omega_1 = \{(x,z) : |x| < \infty, -h_1 \le z < 0\}$ and the upper fluid layer of $\Omega_2 = \{(x,z) : |x| < \infty, 0 \le z \le h_2\}$.

Dimensionless values were introduced using the characteristic L , the maximal free surface elevation a , density of the lower fluid ρ_1 , the acceleration of the gravity g .

The solutions of the nonlinear problem (1) is determined using the method of multiple scale expansions up to the third-order approximation

$$\eta(x,t) = \sum_{n=1}^{3}\alpha^{n-1}\eta_n(x_0, x_1, x_2, t_0, t_1, t_2) + O(\alpha^3) ,$$

$$\eta_0(x,t)=\sum_{n=1}^{3}\alpha^{n-1}\eta_{0n}(x_0,x_1,x_2,t_0,t_1,t_2)+O(\alpha^3)\,, \qquad (2)$$

$$\phi_j(x,z,t)=\sum_{n=1}^{3}\alpha^{n-1}\phi_{jn}(x_0,x_1,x_2,z,t_0,t_1,t_2)+O(\alpha^3), \qquad j=1,2\,,$$

where α is small dimensionless parameter characterizing the steepness ratio of the wave $x_n=\alpha^n x$, $t_n=\alpha^n t$.

2. Evolution equations of the wave-packets envelopes. Substituting (2) into (1) and equating coefficients of like powers of α yields to three linear problems [31].

The solutions of the first-order problem are in the form

$$\eta_1=Ae^{i\theta}+\bar{A}e^{-i\theta}\,,$$

$$\phi_{11}=-\frac{i\omega}{k}(Ae^{i\theta}-\bar{A}e^{-i\theta})\frac{ch(k(h_1+z))}{sh(kh_1)}\,, \qquad (3)$$

$$\phi_{12}=\frac{i\omega}{k}\left(\frac{\omega^2\,sh(k(h_2-z))-(k+T_0k^3)\,ch(k(h_2-z))}{\omega^2\,ch(kh_2)-(k+T_0k^3)\,sh(kh_2)}\right)\left(Ae^{i\theta}-\bar{A}e^{-i\theta}\right),$$

$$\eta_{01}=\left(\frac{\omega^2}{\omega^2\,ch(kh_2)-(k+T_0k^3)\,sh(kh_2)}\right)\left(Ae^{i\theta}+\bar{A}e^{-i\theta}\right),$$

where $\bar{A}(x_1,x_2,t_1,t_2)$ is the complex conjugate of the complex envelope $A(x_1,x_2,t_1,t_2)$, $\theta=kx_0-\omega t$, k is the wave number and ω is the wave frequency of the centre of the wave-packets.

It is easy to see the connection between the envelope of internal wave-packet and envelope wave-packet on the surface of contact.

$$A^0=\frac{\omega^2}{\lambda}A\,, \qquad (4)$$

where $\lambda=\omega^2\,ch(kh_2)-(k+T_0k^3)\,sh(kh_2)$.

The dispersion relationship is

$$\omega^2\,cth(kh_1)+\rho\omega^2\left(\frac{\omega^2-(k+T_0k^3)\,cth(kh_2)}{\omega^2\,cth(kh_2)-(k+T_0k^3)}\right)=(1-\rho)k+Tk^3\,. \qquad (5)$$

Thus, the interface and free surface is linearly stable or unstable depending on whether k is larger or smaller then the cut-off wave number $k_c=\left[(1-\rho)/T\right]^{1/2}$. If surface tension is absent then region of linear stability and linear instability are separated by $\rho=1$.

The solution of the second linear approximation and solvability condition are presented in [31]. Expressions for unknown coefficients were obtained in packages of computer algebra by character transformations, they have cumbersome analytical form.

Substituting the solution of the first linear approximation (1) and obtaining solutions of the second linear approximation in the linear approximation of the third order [31] we obtained the solvability condition. Since the homogeneous part of the third-order problem has a non-trivial solution, the inhomogeneous third-order problem has a solution if, and only if, a solvability condition is satisfied

$$W_1 A_{,t_2} + W_2 A_{,x_2} + W_3 A_{,x_1 x_1} + W_4 A^2 \overline{A} = 0 , \tag{6}$$

$W_i \left(i = \overline{1,4} \right)$ are the coefficients depended on $\left(k, \rho, \omega, h_1, h_2, T, T_0 \right)$.

The group speed $\omega' = d\omega / dk$ was found for simplification of the second order problem solvability condition to the form

$$A_{,t_1} + \omega' A_{,x_1} = 0 , \tag{7}$$

The solvability condition (6) of the third order problem is in the form

$$A_{,t_2} + \omega' A_{,x_2} - 0.5\omega'' A_{,x_1 x_1} = I A^2 \overline{A} , \tag{8}$$

where $\omega'' = d^2\omega / dk^2$.

The evolution equations of envelopes of wave-packets on the fluid interface and the free surface were obtained after adding (8) multiplied at α^2 and (7) multiplied at α

$$A_{,t} + \omega' A_{,x} - 0.5\omega'' A_{,xx} = \alpha^2 I A^2 \overline{A} \tag{9}$$

$$A^0_{,t} + \omega' A^0_{,x} - 0.5\omega'' A^0_{,xx} = \alpha^2 I_0 (A^0)^2 \overline{A}^0 ,$$

where $I_0 = \dfrac{\lambda^2}{\omega^4} I$, detailed analysis of the value $\dfrac{\lambda}{\omega^2}$ has performed in the [31].

The both evolution equations (9) are in the form of the nonlinear Schrödinger equation that is similarly to the results obtained for other hydrodynamic systems [24–30].

3. Analysis of the stability of wave-packets. According to [28, 30], equation (9) has a solution that depends only on time

$$A = a \exp(i\alpha^2 a^2 \omega^{-1} I t) , \tag{10}$$

$$A^0 = a \exp(i\alpha^2 a^2 \omega^{-1} I_0 t) .$$

Then the linear stability conditions on the contact surface and on free surface are of the form

$$I\omega'' > 0 , \qquad I_0 \omega'' > 0 ,$$

it is similar to the results obtained previously [28–30].

As in previous articles [28–30] the conditions of the instable of wave-packets on the interface and free surface are in the form

$$\mathrm{I}\omega'' > 0, \qquad \mathrm{I}_0\omega'' > 0.$$

Curves defined by the equations

$$\mathrm{I}\omega'' = 0 \text{ (curves "1" and "3")}, \quad \mathrm{I}\omega'' \to \infty \text{ (curves "2" and "4")},$$

are presented in Fig. 1. The curves were constructed in a coordinate system (ρ, k), the range of wave numbers that was considered is $0 \le \mathrm{k} \le 2.5$ for different values of thickness of lower fluid layer $\mathrm{h}_1 \in \{1, 1.73, 2.23, 10\}$ and fixed value of $\mathrm{h}_2 = 1$ and in the case of absence of surface tension $\mathrm{T} = 0$, $\mathrm{T}_0 = 0$. The region of linear instability is separated by $\rho = 1$, thus curve with index "3" is the same vertical line. The curve "4" is contained at region of linear instable. Thereby, only curves "1" and "2" define the borders of regions of nonlinear modulational stability (MS) and modulational instability (MI), which are alternated.

If $\mathrm{h}_1 = 10$ the four curves that separate region of nonlinear stability and unstable in 7 regions were found. There are two regions of modulational stability for gravitational and capillary waves in the case of $\rho < 1$ (Fig. 1 a).

Reduction of the thickness h_1 to the value $\mathrm{h}_1 = 2.23$ leads to such result: the curve "1" was raised up, and the curve "2" was rectified near the intersection with the axis of ordinates. The curve "4" was removed up from the axis ρ (Fig. 1 b). Thus, the region of modulational stability has covered physically important region (gravitational and capillary waves).

If $\mathrm{h}_1 = 1.73$ the curve "2" becomes even smoother. In this case three regions of modulational instability have merged (Fig. 1 c). Also, curve "4" is further away from the axis ρ. In the case $\mathrm{h}_1 = 1$ the curve "1" rose up and moves to the left, thus the region of unstable increased (Fig. 1 d).

It was found [31] that the value λ / ω^2 is not equal to zero and it don't tend to infinity, so, in view of $\mathrm{I}_0 = \mathrm{I}\lambda^2 / \omega^4$ the diagrams of stability for wave packets on the interface will be the same as for the free surface.

The studying of dispersion equation (5) in [31] revealed the presence of two pairs of roots of the equation. The first pair of roots is similar to previously obtained during the investigation of hydrodynamic systems "half-space – half-space," "half-space – layer", "layer – layer". In this article we investigated the stability of the wave-packet due to just this pair of roots.

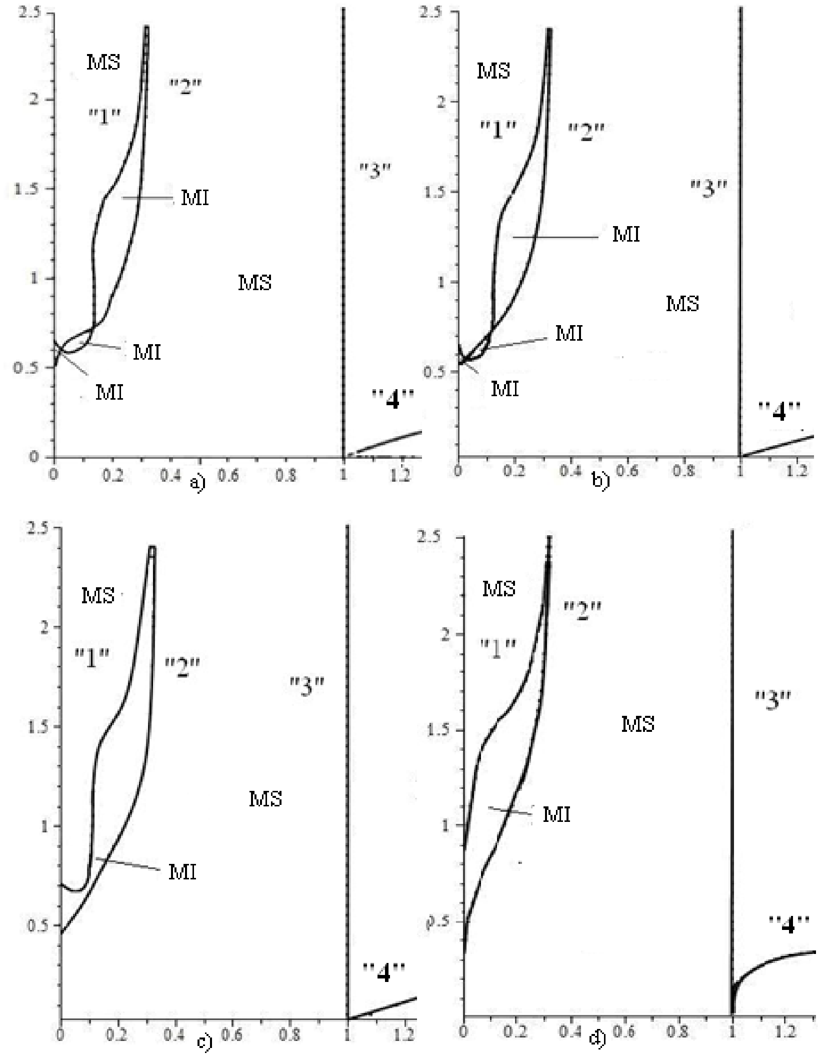

Fig. 1. Diagrams of stability: $T = 0$, $T_0 = 0$, $h_2 = 1$:
a) $h_1 = 10$, b) $h_1 = 2.23$, c) $h_1 = 1.73$, d) $h_1 = 1$

4. Conclusions. The stability of wave packets propagation on the contact surface and free surface of hydrodynamic system "layer with rigid bottom – layer with free surface" was investigated. The diagrams for nonlinear modulational stability for different thicknesses of lower layer were constructed. The presence of large regions of nonlinear modulational stability for capillary and gravitational waves for different ratios of density and different thicknesses of the two fluid layers was obtained. It was noted that the region of modulational nonlinear instability of the wave packet increased with decreasing of thickness of the lower layer.

REFERENCES

1. Choi W., Camassa R. Weakly nonlinear internal waves in a two-fluid system // J. Fluid Mech. – 1996. – v.313. – P. 83–103.
2. Holyer J.Y. Large amplitude progressive interfacial waves // J. Fluid Mech. – 1979. – v.118(3). – P. 433–448.
3. Bourtos Y.Z., Abl-el-Malex M.B., Tewfick A.H. A format expansion procedure for the internal solitary wave problem in a two-fluid system of constant topography // Acta Mechanica. – 1991. – v.88. – P. 172–197.
4. Nayfeh A.H. Nonlinear propagation of wave-packets on fluid interface // Trans. ASME., Ser. E. – 1976. – v.43, N4. – P. 584–588.
5. Hasimoto H., Ono H. Nonlinear modulation of gravity waves // J. of the Phys. Soc. of Japan. – 1972. – v.33. – P. 805–811.
6. Segur H., Hammack J.L. Soliton models of long internal // J. Fluid Mech. – 1982. – v.118. – P. 285–304.
7. Yuen H.C., Lake B.M. Nonlinear dynamics of deep-water waves // Advances in Appl. Mech. New York, London. – 1982. – v. 22. – P. 33–45.
8. Ablowitz M., Segur H. Solitons and the Inverse Scattering Transform. – Moscow: Mir, 1987. – 485 p. (in Russian)
9. Whitham J. Linear and nonlinear waves. – Moscow: Mir, 1977. – 622 p. (in Russian)
10. Bhatnagar P.L. Nonlinear waves in one-dimensional dispersive systems. – Oxford: Clarendon Press, 1979.
11. Lamb J. Introduction to the theory of solitons. – Moscow: Mir, 1983. – 294 p. (in Russian)
12. Selezov I.T., Korsunsky S.V. Wave propagation along the interface between the liquid metal and electrolyte // Proc. Intern. Conf. "MHD Processes to Protection of Environment". Part 1-i. – 1992. – P. 111–117.
13. Selezov I.T., Huq P. Interfacial solitary waves in a three-fluid medium with sourth // 2nd Europ. Fluid Mech. Conf. Abstracts and Papers. – Warsaw, 1994. – 250 p.
14. Bontozoglou V. Weakly nonlinear Kelvin–Helmholz waters between fluids of finite depth // Int. J. Multiphase Flow. – 1991. – v.17, N4. – P. 509–518.
15. Dias F., Kharif Ch. Nonlinear gravity and capillary-gravity waves. Part 7. Importance of surface tension Effects // Annu. Rev. Fluid Mech. – 1999. – N31. –P. 301–346
16. Duncan J.H. Spilling breakers // Annu. Rev. Fluid Mech. – 2001. – 33. – P. 519–547.
17. Baker G.R. Meiron D.I., Orszag S.A. Generalized vortex methods of free-surface flow problems // J.Fluid Mech. – 1982. – v.123. – P. 477–501.

18. Bourtos Y.Z., Abl-el-Malex M.B., Tewfick A.H. A format expansion procedure for the internal solitary wave problem in a two-fluid system of constant topography // Acta Mechanica. – 1991. – v.88. – P. 172–197.
19. Chen Y., Liu P.L.-F. The unified Kadomtsev – Petviashvily equation for interfacial waves // J. Fluid Mech. – 1995. – v.228. – P. 383–408.
20. Stamp A.P., Jacka M. Deep-water internal solitary waves // J. Fluid Mech. – 1995. – 305. – P. 347–371.
21. Trulsen K. Wave kinematics computed with the nonlinear Schroedinger method for deep water // Trans. ASME. – 1999. – N121. – P. 126–130.
22. Avramenko O.V., Selezov I.T. Nonlinear wave propagation in a fluid layer based on a semi-infinite fluid // Reports of NASU. – 1997. – N 10. – P. 61–66. (in Russian)
23. Selezov I.T., Avramenko O.V. Nonlinear wave propagation at small frequencies // Theor. and Appl. Mechanics. – 2000. – N 31. – P. 151–157. (in Russian)
24. Selezov I.T., Avramenko O.V. The evolution equation of the third order for the nonlinear wave-packets at near-critical wave numbers // Dynamical Systems. – 2001. – N 17. – P. 58–67. (in Russian)
25. Selezov I., Avramenko O. Some features of nonlinear wave propagating in two-layer fluid // Geophysical Research Abstracts, The 26-th General Assembly of the European Geophys. Soc. – 2001. – N3. – 8102.
26. Selezov I.T., Avramenko O.V. The evolution of nonlinear wave packets in the hydrodynamic system of "layer – a half-space" with surface tension // Math. methods and physical and mechanical fields. – 2001. – v. 44, N 2.– P. 113–122. (in Russian)
27. Selezov I., Avramenko O., Kharif C., Trulsen K. Higher asymptotic approximations for nonlinear internal waves in fluid // Int. Conf. "Nonlinear Partial differential equations." Book of abstracts. Donetsk. – 2001. – P. 105–106.
28. Avramenko O.V., Selezov I.T. The stability of the wave packets in stratified hydrodynamic systems with surface tension // Applied Hydromechanics. – 2001. – N4. – P. 38 – 46. (in Russian)
29. Selezov I.T., Avramenko O.V., Hurtovyy Y.V. Some features of the wave propagating in the two–layer fluid // Applied Hydromechanics. – 2005. – v. 7 (79), N 1. – P. 80–89. (in Russian)
30. Selezov I.T., Avramenko O.V., Hurtovyy Y.V. The nonlinear stability of the wave packets propagation in the two-layer hydrodynamic system // Applied Hydromechanics. – 2006. – v. 8(90), N4. – P.60–65. (in Russian)
31. Selezov I.T., Avramenko O.V., Hurtovyy Y.V., Naradovy V.V. The nonlinear interaction of internal and external gravity waves in two-layer fluid with free // Math. methods and physical and mechanical fields.– 2009. – v.52, N1. – P. 72–83. (in Russian)
32. Selezov I.T., Avramenko O.V., Naradovy V.V. Some features of nonlinear wave propagating in two-layer fluid with free surface // Dynamical Systems. – 2011. – v. 1(29), N1. – P. 53–68. (in Russian)