SOLUTIONS OF MAXWELL EQUATIONS OF SPECIAL KIND. CHARGE, SPIN AND MASS IN CLASSICAL ELECTRODYNAMICS

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The modern physics has created an impression, that classical electrodynamics is not able to resolve it's inner contradictions. Those, which exist from the very beginning of it's existence as a science. There are fields of elementary particles in it's foundation, but this charged elementary particles are foreigners in the theory of the electromagnetic field. There were lots of attempts to build classical micro electrodynamics, but not all of the contradictions have been successfully removed. The most famous attempts are: Lorentz' electron theory, Abraham's theory of the absolutely rigid electron, Born–Infeld's theory, Mi's potential theory ([1], [2], [3]). The first problem, and also the simplest one, is that the energy of an electric field of a point charge is infinite. This problem can be resolved by assuming, that Coulomb law is correct only outside of some sphere. On this assuming the notion of classical radius of electron was obtained, and electric field is changing sharply on this sphere, determining the surface distribution of charge. If to look from the mathematical physic's point of view, this approach means that we take different solutions of the Laplace equation for the potential of an electric field inside and outside of the electron. Second problem deals with the hypotheses that electric field is a "source" of electron's mass. This is so called "4/3 paradox", which means that the $E = mc^2$ relation does not work with an electric field. In his wonderful book [4], Becker showed the cause of this paradox very clearly and also suggested the way to solve it, but his hypotheses of the electron's rigidity, as we now know, leads to the fact, that electron is a point particle, and that is in conflict with the basic assumption. That is why his methods are not applicable for the micro level. His approach has been discussed and applied to macro objects many times during the last years [6], and some interesting results were obtained. Thus, there is no solution found for "4/3 paradox" for micro level in electrodynamics. Third problem is that electron and other elementary particles do posses an intrinsic moment of momentum and magnetic moment (spin). There is no answer for this given by the classical theory.

In this work we suggest the approach, which allows us to build electromagnetic models of electron and other charged leptons without crossing the limits of classical electrodynamics, and by using the minimal assumption. Or, in other words, mass, charge and spin do have an electromagnetic origin according to this theory, and there are no contradictions from those, that are listed above. Analysis shows us, that we need to decline the notion of electrical charge as the source of electromagnetic field. That leads us to the necessity of discussing the Maxwell system of homogeneous equations in a free space. There are two possible ways of building the theory: 1. Geometrical approach, when one assumes, that free space–time doesn't appear to be a Minkowski plane space–time; 2. Physical approach, when one assumes that vacuum is an inhomogeneous medium. Geometrical approach was published in [7]. We are suggesting the second approach in this work. The first report on this theme was made during the conference [8].

1. Formulation of the problem. Position of any point in a 3-dimensional space is defined by the radius-vector $\vec{\mathbf{x}} = \sum_{i=1}^{3} x_i \vec{\mathbf{e}}_i$, where $\vec{\mathbf{e}}_i$ are the unit vectors of the rectangular coordinate system. Lets denote $r = |\vec{\mathbf{x}}|$, $\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \sum_{i=1}^{3} E_i \vec{\mathbf{e}}_i$ is electric field strength, $\vec{\mathbf{B}}(\vec{\mathbf{x}}) = \sum_{i=1}^{3} B_i \vec{\mathbf{e}}_i$ is magnetic induction, c is velocity of light, ε_0, μ_0 are electric and magnetic permeabilities of vacuum in classical electrodynamics.

We are assuming in our model, that permittivity of vacuum equals to $\varepsilon_0 \varepsilon(\vec{\mathbf{x}})$, and magnetic permeability equals $\mu_0 \mu(\vec{\mathbf{x}})$, and the following relation

$$c^2 \varepsilon_0 \varepsilon(\vec{\mathbf{x}}) \mu_0 \mu(\vec{\mathbf{x}}) = 1 \tag{1}$$

is correct in every point of space. This means, that $\varepsilon(\vec{\mathbf{x}})\mu(\vec{\mathbf{x}}) = 1$. This assumption is based upon the following considerations: firstly, in this case Minkowski material relations between the induction and the strength of magnetic field look the same in all of inertial reference systems, that is natural in vacuum; secondly, Minkowski and Abraham's energy–momentum tensors of electromagnetic field become equal under this condition; and, thirdly, equations of electrostatics and magneto statics appear to be bounded by the function $\varepsilon(\vec{\mathbf{x}})$. Namely, in this case Maxwell system of static homogeneous equations in a free space looks like the following:

$$\begin{cases} (\nabla, \varepsilon \vec{\mathbf{E}}(\vec{\mathbf{x}})) = 0 \\ [\nabla, \vec{\mathbf{E}}(\vec{\mathbf{x}})] \quad = 0 \end{cases}, \qquad \begin{cases} (\nabla, \vec{\mathbf{B}}(\vec{\mathbf{x}})) \quad = 0 \\ [\nabla, \varepsilon \vec{\mathbf{B}}(\vec{\mathbf{x}})] = 0 \end{cases}. \tag{2}$$

By differentiating in (2.1) and in (2.4), let us rewrite these equations in the following form:

$$(\nabla, \vec{\mathbf{E}}(\vec{\mathbf{x}})) = \rho(\vec{\mathbf{x}}) / \varepsilon_0 , \tag{3}$$

$$[\nabla, \vec{\mathbf{B}}(\vec{\mathbf{x}})] = \vec{\mathbf{j}}(\vec{\mathbf{x}}) \big/ \left(\varepsilon_0 c^2 \right) , \tag{4}$$

where

$$\rho(\vec{\mathbf{x}}) = -\varepsilon_0 (\nabla \ln |\varepsilon(\vec{\mathbf{x}})|, \vec{\mathbf{E}}(\vec{\mathbf{x}})) , \tag{5}$$

$$\vec{\mathbf{j}}(\vec{\mathbf{x}}) = -\varepsilon_0 c^2 [\nabla \ln |\varepsilon(\vec{\mathbf{x}})|, \vec{\mathbf{B}}(\vec{\mathbf{x}})] . \tag{6}$$

Let's name $\rho(\vec{\mathbf{x}})$ as charge's volume density, and $\vec{\mathbf{j}}(\vec{\mathbf{x}})$ as current's volume density.

Thus, $\vec{\mathbf{E}}(\vec{\mathbf{x}})$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}})$ fields are the sources of charge and current respectively. The nature of current density is not bounded to the motion of charge density accordingly to this notion. And the total current of any globe layer having center in the origin of coordinates equals null.

Total energy of solution of system (2) exists as the sum of electric field energy and magnetic field energy:

$$W = W_E + W_B,$$

where

$$W_E = 0{,}5\varepsilon_0 \int_{R_3} \varepsilon(\vec{\mathbf{x}})(\vec{\mathbf{E}}(\vec{\mathbf{x}}),\vec{\mathbf{E}}(\vec{\mathbf{x}}))dx, \qquad (7)$$

$$W_B = 0{,}5\varepsilon_0 c^2 \int_{R_3} \varepsilon(\vec{\mathbf{x}})(\vec{\mathbf{B}}(\vec{\mathbf{x}}),\vec{\mathbf{B}}(\vec{\mathbf{x}}))dx, \qquad (8)$$

and the integrals are taken on the whole 3-dimensional space, and $dx = dx_1 dx_2 dx_3$.

So, our current task is: to find the solution of the system (2), which satisfy the following conditions:

1. Field's values are finite in all the points of space.
2. Electromagnetic field energy is finite.
3. Function $\varepsilon(\vec{\mathbf{x}}) \to 1$ when $|\vec{\mathbf{x}}| \to \infty$.

2. **Elementary solutions of Maxwell equations.** The condition of energy finitude means, that $\vec{\mathbf{E}}(\vec{\mathbf{x}}),\ \vec{\mathbf{B}}(\vec{\mathbf{x}}) \in L_{2,\varepsilon}(R^3)$. Because of the homogeneity of the equations we can use method of variables separation to provide the research.

Let

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = -\nabla\varphi(\vec{\mathbf{x}}), \qquad (9)$$

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}) = -\frac{1}{\varepsilon(r)}\nabla\psi(\vec{\mathbf{x}}), \qquad (10)$$

than (2.2), (2.4) equations are fulfilled, and solution of the problem is reduced to the study of the following equations:

$$\left(\nabla,\ \varepsilon(r)\nabla\varphi(\vec{\mathbf{x}})\right) = 0, \qquad (11)$$

$$\left(\nabla, \frac{1}{\varepsilon(r)}\nabla\psi(\vec{\mathbf{x}})\right) = 0. \qquad (12)$$

Elementary solutions (11), (12) have the following form:

$$\varphi_n(\vec{\mathbf{x}}) = H_n(r)Y_n^E(\vartheta,\phi), \qquad (13)$$

$$\psi_n(\vec{\mathbf{x}}) = F_n(r) Y_n^B(\vartheta,\phi), \tag{14}$$

where $Y_n^E(\vartheta,\phi)$, $Y_n^B(\vartheta,\phi)$ are spherical functions and $F_n(r)$, $H_n(r)$ are radial functions, and they suppose to be the solutions of the corresponding radial differential equations, namely:

$$r^2 F_n''(r) + r\left(2 - r\frac{\varepsilon'(r)}{\varepsilon(r)}\right) F_n'(r) - n(n+1) F_n(r) = 0, \tag{15}$$

$$r^2 H_n''(r) + r\left(2 + r\frac{\varepsilon'(r)}{\varepsilon(r)}\right) H_n'(r) - n(n+1) H_n(r) = 0. \tag{16}$$

The variables r,ϑ,ϕ in (13), (14) define a spherical coordinate system

$$x_1 = r\cos\phi\sin\vartheta,\ x_2 = r\sin\phi\sin\vartheta,\ x_3 = r\cos\vartheta .$$

The solutions of radial equations (15), (16) are the most interesting. Lets note, that for the equations we've obtained there are two kinds of symmetry:

– explicit symmetry, which directly arises from the form of equations (11), (12) (or (15), (16)). This means, that if we replace $\varepsilon(r)$ with $\varepsilon^{-1}(r)$, these equations would change over, alias representations for electric and magnetic fields would change over

– implicit symmetry, which gives us simple relation between electric and magnetic fields. Namely, the following is valid:

Statement 1. *The relation* $F_n(r) = \varepsilon(r) H_n(r)$ *between the solutions of radial equations (15), (16) is valid then and only then, when* $\varepsilon(r) = e^{\pm\lambda/r}$.

This function arises independently in other parts of our theory, so we will consider that $\varepsilon(r) = e^{\lambda/r}$, $\lambda \ge 0$ in what follows, unless otherwise indicated.

Statement 2. *The solution of the equation(15), which secures the finiteness of the integral of magnetic field energy, has the following form:*

$$F_n(r) = \begin{cases} c_n \left(r/\lambda\right)^n M(-n;-2n;\lambda/r), & 0 \le r \le a \\ b_n \left(\lambda/r\right)^{n+1} M(1+n;2+2n;\lambda/r), & a < r < \infty \end{cases}. \tag{17}$$

$F_n(r) \notin C^1[0,\infty)$.

In here $M(a,b,x)$ is a degenerate hypergeometric function, Kummer function ([9]). While obtaining (17), also were obtained the following finite species representations of Kummer function:

$$M(1+n;2+2n;\xi) = \sum_{m=n}^{2n} (m+1)! \binom{m}{n} \binom{2n}{m+1} \left(1 - (-1)^m e^{\xi}\right) \xi^{-m-1}$$

$$M(-n;-2n;\xi) = \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \binom{2n}{k}^{-1} \xi^k$$

Statement 3. *Radial equation (16) for an electric field has the solutions, which secure the finiteness of an electric field energy, and they have the following form:*

– $n = 0,\ H_0(r) = c_0(1 - e^{-\lambda/r}) \in C^\infty[0,\infty)$;

– $n > 0$,

$$H_n(r) = \begin{cases} c_n \left(r/\lambda\right)^n e^{-\lambda/r} M(-n;-2n;\lambda/r), & 0 \le r \le a \\ b_n \left(\lambda/r\right)^{n+1} e^{-\lambda/r} M(1+n;2+2n;\lambda/r), & a < r < \infty \end{cases}. \quad (18)$$

$$H_n(r) \notin C^1[0,\infty)$$

3. Energy and mass of an electromagnetic field. It is obvious, that it would be sufficient to investigate the energy of elementary solutions, because of the orthogonality of spherical functions. Behavior of elementary solutions in null and on the infinity provides us with the following:

Statement 4. *The following formula is valid for the energy of an elementary solution for an axisymmetric magnetic field with one point of break:*

$$W_{B_n} = \frac{\pi\varepsilon_0 c^2 a^2}{(2n+1)\varepsilon(a)}\left[(F_n{}^2(a_-))' - (F_n{}^2(a_+))'\right]. \quad (19)$$

In here by $F_n(a_-),\ F_n(a_+)$ we denote limiting values on the left and on the right

Also the next statement is valid.

Statement 5. *The following formula is valid for the energy of an elementary solution for an axisymmetric electric field with one point of break:*

$$W_{E_n} = \frac{\pi\varepsilon_0 a^2 \varepsilon(a)}{(2n+1)}\left[(H_n{}^2(a_-))' - (H_n{}^2(a_+))'\right]. \quad (20)$$

The energy of a centrally symmetric electric field is defined by the expression

$$W_E = 0{,}5 q \varphi_0(0), \quad (21)$$

where the potential of a centrally symmetric field has the form of

$$\varphi_0(r) = \frac{q}{4\pi\varepsilon_0}\int_r^\infty \frac{dt}{t^2\varepsilon(t)}, \quad (22)$$

and equals to the Coulomb potential outside the sphere, *q* – integration constant, which defines the full electric charge of a field. Energy formulas are correct for an arbitrary function $\varepsilon(r)$, which has the asymptotic form we've defined above.

Electric field of electron is centrally symmetric, so from here we will discuss only centrally symmetric electric field and axisymmetric magnetic field.

By using the energy–momentum tensor of an electromagnetic field and with the help of Lorentz transformation one can prove

Statement 6. *In any inertial reference system a minimum of the energy of electromagnetic field is reached, when the axis of magnetic field is parallel to the velocity of field's motion.*

The corollary fact of this is

Statement 7. *In any inertial reference system the axis of magnetic field is parallel to the velocity of field's motion.*

At the end of 19 and at the beginning of 20 centuries physicists were trying to construct a theory, which would have allowed to consider an electron mass as equal to mass of his electromagnetic field, just until it became clear, that it is impossible. As we know, a rest mass m_0 (inert mass) is a proportionality coefficient between momentum and velocity when velocity tends to zero. If we would consider to identify electron with electric field of its charge, than for the energy W_E of the electric field the inequality $W_E < m_0c^2$ is valid for any charge distribution "inside" the electron (see [10], [11]). This inequality was translated as if some part of electron mass has no electromagnetic origin. But if we have a non-nil magnetic field, we can say that $W_E + W_B = m_0c^2$, which means that the missing part of rest mass of the solutions of system (2) has magnetic origin.

Statement 8. *Among the solutions of Maxwell equations (2) there are solutions with a centrally symmetric electric field and axisymmetric non-nil magnetic field, and they state, that all the field's inertial mass has electromagnetic origin.*

Definition. *Lets term the solution of Maxell equations (2) as physical, if the following relation is correct for it* $W = mc^2$, *where* W *is the energy of this solution,* m *is its inertial mass.*

From this point of view a centrally symmetric electric field, which appears to be a solution of system (2), if magnetic field is zero-order, is not a physical solution. Vice versa, axisymmetric magnetic field is not a physical solution of system (2), if electric field is zero-order.

Thus, *"4/3 paradox" is being solved in a natural way without crossing the limits of electromagnetic theory.*

4. Moment of momentum and magnetic moment. The existence of a magnetic field allows us to answer a question about the existence of electron's intrinsic moment of momentum and magnetic moment positively. A non-nil contribution to these characteristics, because of the orthogonality of Legendre polynomials, is given only by the field

$$\vec{\mathbf{B}}_1(\vec{\mathbf{x}}) = -\frac{1}{\varepsilon(r)}\nabla(F_1(r)\cos\theta) . \tag{23}$$

Thus in further we will discuss an electromagnetic field, which is defined by the centrally symmetric electric field

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{\mathbf{x}}}{r^3\varepsilon(r)} \tag{24}$$

and by the axisymmetric magnetic field (23), and indexes will be omitted.

A moment of momentum is defined by the formula $\vec{\mathbf{P}} = (0, 0, P_3)$, where

$$P_3 = \varepsilon_0 E_0 \frac{8\pi}{3}\int_0^\infty \frac{F(r)}{\varepsilon(r)}dr = -\frac{qa^2}{3\varepsilon(a)}\left[F'(a_+) - F'(a_-)\right]. \tag{25}$$

This equality is valid for any acceptable $\varepsilon(r)$. The jump of a normal component of magnetic field on a sphere S_a is connected with a moment of momentum by the relation

$$\left(\vec{\mathbf{n}}, \vec{\mathbf{B}}_+ - \vec{\mathbf{B}}_-\right)\Big|_{r=a} = \frac{3}{qa^2}\left(\vec{\mathbf{P}}, \vec{\mathbf{n}}\right)$$

The presence of the jump of a normal component of magnetic field means, that there is a distribution of magnetic charge specified on the sphere S_a, with a surface density

$$\sigma(\theta) = \frac{3\varepsilon_0 c}{qa^2}(\vec{\mathbf{P}}, \vec{\mathbf{n}}), \tag{26}$$

and it is evident, that the full magnetic charge on S_a is null, so thus we have a dipole distribution of a surface magnetic charge. *Full magnetic charge with a definite sign does not depend on* a *and equals*

$$q_m = 3\pi\varepsilon_0 cP_3/q = 0.75\varepsilon_0 ch/q = \alpha q,$$

where q is full electric charge of a field, h is Planck constant, $\alpha = 3\pi\varepsilon_0 cP_3/q^2 \cong 51{,}3885$.

A conversion of moment of momentum from a stationary reference system $\mathbf{K}(\vec{\mathbf{x}}, t)$, where $P_1 = (q/4\pi)I_1 \cos\phi_0 \sin\vartheta_0$, $P_2 = (q/4\pi)I_1 \sin\phi_0 \sin\vartheta_0$, $P_3 = (q/4\pi)\, I_1 \cos\vartheta_0$, into the system $\mathbf{K}'(\vec{\mathbf{x}}', t')$, which is moving with the velocity of $\vec{\mathbf{V}} = (0, 0, V)$ with respect to the system $\mathbf{K}(\vec{\mathbf{x}}, t)$, is defined by the following formulas:

$$P_1' = P_1\Big/\sqrt{1-\beta^2} + t'\varepsilon_0\beta^2 I_2 \sin\phi_0 \cos\vartheta_0 \sin\vartheta_0,$$

$$P_2' = P_2\Big/\sqrt{1-\beta^2} - t'\varepsilon_0\beta^2 I_2 \cos\phi_0 \cos\vartheta_0 \sin\vartheta_0, \; P_3' = P_3,$$

from which the following emerges:

Statement 9. *A projection of moment of momentum onto the direction of velocity of motion is retained.*

By I_1, I_2 in last formulas we mean some integral constants.

Let's note, that moment of momentum in a moving reference system can be represented in a form $\vec{\mathbf{P}}' = \vec{\mathbf{P}}'_s + \vec{\mathbf{P}}'_{orb}$, where $\mathbf{P}'$ is an intrinsic moment of momentum (spin) of an electromagnetic field. Vector $\mathbf{P}'$ is an orbital moment of momentum, and emerges due to the noninertial movement of system and is defined only by the magnetic field, and that is natural because magnetic field is axisymmetric and electric field is centrally symmetric. It is evident, that equality $\mathbf{P}', \mathbf{P}'$ is correct, which means that spin and orbital moment are orthogonal always. Hence, $\left|\vec{\mathbf{P}}'\right| = \sqrt{\left|\vec{\mathbf{P}}'_s\right|^2 + \left|\vec{\mathbf{P}}'_{orb}\right|^2}$. Let's also note, that orbital moment exists in a plane, which is perpendicular to the direction of motion, and is tangential to the circle with center on the axis Ox_3 in any moment of time, and also that it broaches the axis of magnetic field so that this axis becomes parallel to the direction of motion. Thus, it would be sharply to refer to this orbital moment as to the orienting moment of momentum. Orienting moment becomes null if: a) the axis of magnetic field is parallel to the velocity, $\vartheta_0 = 0;\ \pi;$ b) the axis of magnetic field is perpendicular to the velocity, $\vartheta_0 = \pi/2$. Case a) corresponds to a minimal value of system energy, alias to a stable state of system and with two opposite directions of spin, $\vec{\mathbf{P}}_s = (0; 0; \pm P_{s3})$. Case b) corresponds to a maximum value of system energy, alias to an unstable state of system, with a zero projection of spin onto the direction of velocity.

Thus, orienting moment restores the axial symmetry of system in a moving reference system. The intrinsic moment of momentum (spin) determines the invariant rotation, alias rotation, which does not change system status.

It is evident, that reference system $\mathbf{K}'(\mathbf{x}'$ would be inertial, only if the condition a) is true. Hence, the following is valid:

Statement 10. *In any inertial reference system spin vector is collinear with a vector of velocity of field's motion, and spin value is constant.*

In natural way we can introduce spin 4-vector, which is always orthogonal to the field's velocity 4-vector. Spin 4-vector is a space-like vector, and its value is constant in inertial reference systems. We've obtained transformations of 3-dimensional intrinsic moment of momentum of a free electromagnetic field while moving from one inertial system to another.

For calculating a full magnetic moment we are to take into account, that we have volumetric stream distribution, surface stream distribution and surface distribution of magnetic charge. After performing all the needed calculations, we obtain that magnetic moment of electromagnetic field equals $\vec{\mathbf{M}} = (0, 0, M_3)$, where $M_3 = \lambda^2 b_1$, and constant b_1 determines the behavior of magnetic field in the neighborhood of infinitely remote point (see expression (17)). Thus, magnetic moment is collinear with the axis of magnetic field.

5. Conclusions. From the researches was obtained an electron model, which has all the characteristics of a real electron. Charge, spin, mass and magnetic moment in the model have an electromagnetic nature. It's not possible to describe this model completely, in details, within the limits of this article. It'll be done in our next article.

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