PROBLEMS OF UNIFORMLY VALID ASYMPTOTICS FOR THE BOLTZMANN EQUATION IN THE HYDRODYNAMICAL LIMIT

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The multiscale method is used to obtain a uniformly valid asymptotic expansion for linearized Boltzmann equation for small Knudsen number. This approach extends Hilbert expansion for large times and permits to get the regular gas-dynamic-type equations. The one-dimensional problem of small disturbances in gas is considered.

The hydrodynamic Navier–Stokes equations are formulated on the basis of continuum mechanics. [1]. On the other hand, the integro-differential Boltzmann equation describes gas according to the atomic point of view [2]. It is well known that fundamental principles of the continuum mechanics and kinetic theory are different. Nevertheless in the limiting case of small Knudsen numbers $Kn = l_s / L_s$ (l_s is the molecular free path and L_s is a typical size) it is possible to use both the kinetic approach and the continuum description and to study the relation between these two approaches. D.Hilbert was the first who used a series expansions in powers of the small Knudsen number and expressed distribution functions in terms of the expansions of the macroscopic variables. However, the simple expansions in powers of a small parameter break down for long times because the perturbations grow with time [3–5]. The classical Chapman–Enskog is too not able to provide the regularity of the expansion and yields unsatisfactory Burnett equations. The instability of the Burnett equations proved by A. Bobylev in [6] confirms an irregularity of the Chapman–Enskog procedure. Bobylev paper was a starting point for many publications on the stability problem for the Burnett and the super–Burnett equations. [7–11]. Notice that the most of them construct methods for regularization at the level of Burnett equations and, as a rule, are based on a combination of the Chapman–Enskog method with moment methods. A detailed analysis of these problems contains in [8]. Generally speaking, the presence of a small parameter permits to apply the systematic methods of asymptotic expansions. For instance, the monograph by Y. Sony [12] is intended to asymptotic analysis of the boundary-value problems of the Boltzmann equation. On the other hand, the perturbation theory proposes some methods that would be well-adapted to study the problem of the long time behaviour [3.4]. One of these methods is the multiscale technique. The first attempt to use it in the considered problem was undertaken in [13]. Unfortunately, the authors didn't take into account the compatibility problem of the expansions [14]. The uniformly valid self-consistent asymptotic expansion with the help of the multiscale technique was obtained in [15]. The considered problem is certainly related with the common problem of convergence to equilibrium for solutions of the Boltzmann equation. This subject is investigated in detail by authors [16]. In the present work the multiscale technique is used for the linearized case to study a time evolution of small perturbations in gas. In particular, the regular Burnett equations are obtained.

Consider the following one-dimensional initial value problem: a dissipative decay of small disturbance in a simple gas at rest in the limiting case of the small Knudsen number $\mathrm{Kn} = \varepsilon$. Confining ourselves to the linear approximation we use the linearized nondimensional Boltzmann evolution equation

$$\begin{gathered} \mathrm{L}\varphi = \varepsilon\left(\frac{\partial\varphi}{\partial t} + c_x\frac{\partial\varphi}{\partial x}\right), \\ \mathrm{L}\varphi = \int f_0(\mathbf{c}_1)\left[\varphi(\mathbf{c}') + \varphi(\mathbf{c}'_1) - \varphi(\mathbf{c}) - \varphi(\mathbf{c}_1)\right] g b\, db\, d\beta\, d\mathbf{c}_1 \end{gathered} \tag{1}$$

We take the scales from the paper [14] and

$$f = f_0\varphi, \quad f_0 = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{c^2}{2}\right) \tag{2}$$

The hydrodynamic values are defined as

$$n = \int f_0\varphi\, d\mathbf{c}, \quad v = \int c_x f_0\varphi\, d\mathbf{c}, \quad 3p = \int c^2 f_0\varphi\, d\mathbf{c}, \quad p = n + T \tag{3}$$

Further, we will construct asymptotic approximation to the solution of (1). As the Boltzmann equation has two essentially different scales l_s and L_s, we apply the multiscale method [3,4]. We suppose that the distribution function depends on auxiliary time variables $t_k = \varepsilon^k t$, as well as ε itself, and assume φ in the form

$$\varphi = \varphi_0(\mathbf{c}, x, t_0, t_1, t_2 \ldots) + \varepsilon\varphi_1(\mathbf{c}, x, t_0, t_1, t_2 \ldots) + \ldots, \tag{4}$$

that implies the related expansions

$$\begin{gathered} n = \sum_0^N n_k, \quad v = \sum_0^N v_k, \quad p = \sum_0^N p_k, \quad T = \sum_0^N T_k, \quad T_k = p_k - n_k \\ n_k = \int f_0\varphi_k\, d\mathbf{c}, \quad v_k = \int c_x f_0\varphi_k\, d\mathbf{c}, \quad 3p_k = \int c^2 f_0\varphi_k\, d\mathbf{c}, \end{gathered} \tag{5}$$

Since t_k are treated as new independent variables, the time derivative is transformed

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + \ldots, \tag{6}$$

Substitution the expansions (4), (6) into (1) gives a sequence of the integral equations

$$L\varphi_k = Q_k \tag{7}$$

where the source terms Q_k are constructed by means of the previous approximations $Q_0 = 0$, $Q_1 = \frac{\partial \varphi_0}{\partial t_0} + c_x \frac{\partial \varphi_0}{\partial x}$, $Q_2 = \frac{\partial \varphi_0}{\partial t_1} + \frac{\partial \varphi_1}{\partial t_0} + c_x \frac{\partial \varphi_1}{\partial x}$, $Q_3 = \frac{\partial \varphi_0}{\partial t_2} + \frac{\partial \varphi_1}{\partial t_1} + \frac{\partial \varphi_2}{\partial t_0} + c_x \frac{\partial \varphi_2}{\partial x}$.

At each step we can find φ_k, provided that the Q_k satisfies the solvability conditions

$$\int \psi_r f_0 Q_k d\mathbf{c} = 0\ , \quad \psi_1 = 1, \quad \psi_1 = c_x\ , \ \psi_3 = c^2 \tag{8}$$

The set (7) differs essentially from the Hilbert procedure and involves arbitrary functions of the time scales t_k. We utilize the resulting freedom at each step of the procedure and impose additional conditions to remove secularities. Our only explicit restrictions will be: if the expansions in (4) are to be valid for a long time, it is necessary to require that higher approximations be small corrections to the first term throughout the time range.

The solutions of (7) can be represented in the form [2]

$$\varphi_k = \alpha_k + \beta_k c_x + \gamma_k c^2 + h_k \tag{9}$$

with particular solution h_k. Arbitrary functions $\alpha_k, \beta_k, \gamma_k$ are defined by the solvability conditions (8) and can be expressed in terms of the values n_k, p_k, v_k.

Since at the lowest order $L\varphi_0 = 0$, we have

$$\varphi_0 = p_0 + c_x v_0 + \left(\frac{c^2}{2} - \frac{5}{2} \right) T_0 \tag{10}$$

The solvability conditions (8) for the second equation from (7) lead to the linearized Euler equations with respect to fast time t_0,

$$\frac{\partial n_0}{\partial t_0} = -\frac{\partial v_0}{\partial x}, \quad \frac{\partial v_0}{\partial t_0} = -\frac{\partial p_0}{\partial x}, \quad \frac{\partial p_0}{\partial t_0} = -\frac{5}{3}\frac{\partial v_0}{\partial x} \tag{11}$$

This set gives the wave equations for v_0 and p_0 and the adiabatic relation

$$\left(\frac{\partial^2}{\partial t_0^2}-\frac{5}{3}\frac{\partial^2}{\partial x^2}\right)v_0=0,\quad \left(\frac{\partial^2}{\partial t_0^2}-\frac{5}{3}\frac{\partial^2}{\partial x^2}\right)p_0=0,$$
$$\frac{\partial s_0}{\partial t_0}=0,\qquad s_0=p_0-\frac{5}{3}n_0=\frac{5}{3}T_0-\frac{2}{3}p_0 \tag{12}$$

Finally, for the Maxwell molecules we have the well known result [2]

$$\varphi_1=p_1+v_1c_x+\left(\frac{c^2}{2}-\frac{5}{2}\right)T_1+h_1\ ,$$
$$h_1=-\frac{2}{5}\lambda\left(\frac{c^2}{2}-\frac{5}{2}\right)c_x\frac{\partial T_0}{\partial x}-\mu\left(c_x^2-\frac{c^2}{3}\right)\frac{\partial v_0}{\partial x}\ ,\ \lambda=\frac{15}{4}\mu \tag{13}$$

where μ и λ are viscosity and heat conduction coefficient accordingly. Thus, the evaluation of φ_1 is complete up to $p_1,\ v_1,\ T_1$. We need now to consider the next approximation. Here the solvability conditions are

$$\frac{\partial n_1}{\partial t_0}+\frac{\partial v_1}{\partial x}=-\frac{\partial n_0}{\partial t_1}\ ,\qquad \frac{\partial v_1}{\partial t_0}+\frac{\partial p_1}{\partial x}=-\left(\frac{\partial v_0}{\partial t_1}-\frac{4}{3}\mu\frac{\partial^2 v_0}{\partial x^2}\right),$$
$$\frac{\partial p_1}{\partial t_0}+\frac{5}{3}\frac{\partial v_1}{\partial x}=-\left(\frac{\partial p_0}{\partial t_1}-\frac{5}{2}\mu\frac{\partial^2 T_0}{\partial x^2}\right). \tag{14}$$

Since s_0 doesn't depend on t_0, from the first and the third equations it follows

$$\frac{\partial s_0}{\partial t_1}-\frac{3}{2}\mu\frac{\partial^2 s_0}{\partial x^2}=0\ ,\quad \frac{\partial}{\partial t_0}\left(s_1+\mu\frac{\partial v_0}{\partial x}\right)=0\ ,\ s_k=p_k-\frac{5}{3}n_k \tag{15}$$

The set (14) leads to inhomogeneous wave equations for values p_1, v_1

$$\frac{\partial^2 v_1}{\partial t_0^2}-a^2\frac{\partial^2 v_1}{\partial x^2}+\frac{3}{2}\mu\frac{\partial^3 s_0}{\partial x^3}=-2\frac{\partial}{\partial t_0}\left(\frac{\partial v_0}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 v_0}{\partial x^2}\right),$$
$$\frac{\partial^2 p_1}{\partial t_0^2}-a^2\frac{\partial^2 p_1}{\partial x^2}=-2\frac{\partial}{\partial t_0}\left(\frac{\partial p_0}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 p_0}{\partial x^2}\right). \tag{16}$$

The terms on the right-hand side of (16) produce secular particular solutions. Hence, the expansions (4), (5) fail for times of order of ε^{-1} unless

$$\frac{\partial p_0}{\partial t_1}=\frac{7}{6}\mu\frac{\partial^2 p_0}{\partial x^2},\qquad \frac{\partial v_0}{\partial t_1}=\frac{7}{6}\mu\frac{\partial^2 v_0}{\partial x^2} \tag{17}$$

Note that, although we have.

$$\frac{\partial^2 p_1}{\partial t_0^2}-a^2\frac{\partial^2 p_1}{\partial x^2}=0\ ,\ \frac{\partial^2 \tilde{v}_1}{\partial t_0^2}-a^2\frac{\partial^2 \tilde{v}_1}{\partial x^2}=0\ ,\ a^2v_1=a^2\tilde{v}_1+\frac{3}{2}\mu\frac{\partial s_0}{\partial x}\ , \tag{18}$$

the equations (14) remain inhomogeneous

$$\frac{\partial \tilde{v}_1}{\partial t_0}+\frac{\partial p_1}{\partial x}=\frac{1}{6}\mu\frac{\partial^2 v_0}{\partial x^2}\ , \qquad \frac{\partial p_1}{\partial t_0}+\frac{5}{3}\frac{\partial \tilde{v}_1}{\partial x}=-\frac{1}{6}\mu\frac{\partial^2 p_0}{\partial x^2} \tag{19}$$

To extend the validity of the expansions (4), (5) up to times $O\left(\varepsilon^{-2}\right)$ we must go over to the third approximation. As above, we obtain the set of macroscopic equations

$$\begin{aligned}
&\frac{\partial n_2}{\partial t_0}+\frac{\partial v_2}{\partial x}+\frac{\partial n_1}{\partial t_1}+\frac{\partial n_0}{\partial t_2}=0\ ,\\
&\frac{\partial v_2}{\partial t_0}+\frac{\partial p_2}{\partial x}+\frac{\partial v_0}{\partial t_2}+\frac{\partial v_1}{\partial t_1}=-\frac{\partial}{\partial x}\int f_0\left(c_x^2-\frac{c^2}{3}\right)h_2 d^3c=\frac{\partial}{\partial x}J_1\ ,\\
&\frac{\partial p_2}{\partial t_0}+\frac{5}{3}\frac{\partial v_2}{\partial x}+\frac{\partial p_0}{\partial t_2}+\frac{\partial p_1}{\partial t_1}=-\frac{2}{3}\frac{\partial}{\partial x}\int f_0 c_x\left(\frac{c^2}{2}-\frac{5}{2}\right)h_2 d^3c=\frac{\partial}{\partial x}J_2\\
&J_1=\mu\frac{4}{3}\frac{\partial \tilde{v}_1}{\partial x}+\mu^2\frac{8}{15}\frac{\partial^2 p_0}{\partial x^2}\ ,\quad J_2=-\mu\frac{\partial \tilde{v}_1}{\partial t_0}-\frac{1}{6}\mu^2\frac{\partial^2 v_0}{\partial x^2}+\frac{3}{2}\mu\frac{\partial s_1}{\partial x}
\end{aligned} \tag{20}$$

An immediate consequence of the set (20) with (12), (15) are the relations

$$\frac{\partial s_0}{\partial t_2}=0\ , \qquad \left(\frac{\partial}{\partial t_1}-\frac{3}{2}\mu\frac{\partial^2}{\partial x^2}\right)\left(s_1+\mu\frac{\partial v_0}{\partial x}\right)=0 \tag{21}$$

From the set (20) solved for v_2 we have

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial t_0^2}-a^2\frac{\partial^2}{\partial x^2}\right)\tilde{v}_2=-2\frac{\partial}{\partial t_0}\left(\frac{\partial \tilde{v}_1}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 \tilde{v}_1}{\partial x^2}+\frac{\partial v_0}{\partial t_2}-\frac{19}{120}\mu^2\frac{\partial^3 p_0}{\partial x^3}\right)\ ,\\
&a^2 v_2=a^2\tilde{v}_2+\frac{3}{2}\mu s_1,\ \ s_1=p_1-\frac{5}{3}n_1
\end{aligned} \tag{22}$$

The terms on the right-hand side of (22) make $\varepsilon^2 v_2$ in (5) the same order as εv_1, when t_0 is as large as $O\left(\varepsilon^{-1}\right)$. Hence, the expression in brackets must vanish

$$\frac{\partial \tilde{v}_1}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 \tilde{v}_1}{\partial x^2}=-\frac{\partial v_0}{\partial t_2}+\frac{19}{120}\mu^2\frac{\partial^3 p_0}{\partial x^3}\ , \tag{23}$$

Since the right-hand side of Eq.(23) gives the particular solution

$$t_1\left(\frac{\partial v_0}{\partial t_2}-\frac{19}{120}\mu^2\frac{\partial^3 p_0}{\partial x^3}\right),$$

the expansion $v_0+\varepsilon v_1$ breaks down for t as large as $O\left(\varepsilon^{-2}\right)$. Therefore, we must put

$$\frac{\partial v_0}{\partial t_2}-\frac{19}{120}\mu^2\frac{\partial^3 p_0}{\partial x^3}=0\ ,\qquad \frac{\partial \tilde{v}_1}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 \tilde{v}_1}{\partial x^2}=0 \tag{24}$$

Solving the set (20) for p_2, we get in the same way the following conditions

$$\frac{\partial p_0}{\partial t_2}-\frac{19}{72}\mu^2\frac{\partial^3 v_0}{\partial x^3}=0\ ,\qquad \frac{\partial p_1}{\partial t_1}-\frac{7}{6}\mu\frac{\partial^2 p_1}{\partial x^2}=0 \tag{25}$$

We apply the method of multiple scales to represent the solution by the leading asymptotic term uniformly valid up to times of the order ε^{-2}. Conditions (12), (17), (21), (24), and (25) express the time derivatives in (6) in terms of space derivatives. Then, accordingly to (6), for the leading term we get the system of linearized macroscopic equations

$$\begin{aligned}
&\frac{\partial s_0}{\partial t}-\frac{3}{2}\mu\varepsilon\frac{\partial^2 s_0}{\partial x^2}=0\ ,\\
&\frac{\partial v_0}{\partial t}=-\frac{\partial p_0}{\partial x}+\frac{7}{6}\varepsilon\mu\frac{\partial^2 v_0}{\partial x^2}+\frac{19}{120}\varepsilon^2\mu^2\frac{\partial^3 p_0}{\partial x^3},\\
&\frac{\partial p_0}{\partial t}=-a^2\frac{\partial v_0}{\partial x}+\frac{7}{6}\varepsilon\mu\frac{\partial^2 p_0}{\partial x^2}+\frac{19}{120}\varepsilon^2\mu^2a^2\frac{\partial^3 v_0}{\partial x^3}
\end{aligned} \tag{26}$$

If we take the perturbation in the form

$$v_0=V\exp\left[i\left(kx-\omega t\right)\right],\ \ p_0=P\exp\left[i\left(kx-\omega t\right)\right],\ \ s_0=S\exp\left[i\left(kx-\omega t\right)\right] \tag{27}$$

then we obtain the system of algebraic equations

$$\begin{aligned}
&\left(\iota\omega+\frac{3}{2}\mu\varepsilon k\right)S=0\ ,\ \left(-\iota\omega+\frac{7}{6}\mu\varepsilon k\right)V+ik\left(1+\frac{19}{120}\varepsilon^2\mu^2k^2\right)P=0\\
&\left(-\iota\omega+\frac{7}{6}\mu\varepsilon k\right)P+ika^2\left(1+\frac{19}{120}\varepsilon^2\mu^2k^2\right)P=0
\end{aligned} \tag{28}$$

From this set it follows that :
either the case $S\neq 0,\ \ P=0,\ \ \ V=0$, or the case $S=0,\ \ P\neq 0,\ \ \ V\neq 0$

The latter case describes a propagation of the sound wave

$$\omega=\delta+i\Delta,\qquad \delta=\pm\left(1+\frac{19}{120}\varepsilon^2\mu^2k^2\right)ka,\qquad \Delta=-\frac{7}{6}\varepsilon\mu k^2 \tag{29}$$

The parameter δ gives the Burnett correction for the sound velocity. The decay decrement Δ includes the Stokes– Kirchhof effect.

Thus for the considered linearized one-dimensional problem we get the regular hydrodynamic- type equations.

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