INVESTIGATION OF STRESS ERROR ESTIMATOR IN ELASTICITY PROBLEMS

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Numerical methods of solving elasticity (and other) problems have already proved their effectiveness. To ascertain this fact one can have a look at a great number of commercial software products providing numerical solution of this kind of problems. Most of them use finite element method (FEM) for its universality and relative simplicity in realisation. However, despite the success of the software, they still have imperfections through unsolved questions in the theory. One of these questions is how to build the best approximation space without increasing the number of degrees of freedom too much, i.e. how to construct an optimal finite element mesh. In the case of the problem with no singularities, it can be a uniform mesh, which can be easily constructed using advancing front method [7], Delaunay triangulation method [11] or isoparametric mapping method [11]. If the problem has some singularities, uniform mesh, obviously, can't be optimal; it has to be refined in some way near the singularity point. This can be done either in the very beginning of the problem solving process (*a priori*), or after finding some approximate solution (*a posteriori*). The last case has more benefits since we can use the information about the solution we have. Moreover, we don't need any preassigned information about the singularities: it also can be retrieved from the approximate solution. Therefore a great amount of scientific works propose different techniques of *a posteriori* error estimation [1, 3, 6, 9, 11, 12], which is an element of crucial importance in mesh refinement process. Nevertheless, still there is no universal effective error estimator. We propose a new way of estimation based on the difference in the solutions given by FEM and BEM (boundary element method). It is well known that the precision of stress is O(h) for FEM [11], and O(h2) for BEM [5] (for linear approximations). This fact serves as a reason to use the difference as an error estimator. To apply this technique, FEM and BEM grids have to be connected in some way. Thus we construct BEM grid as a "trace" of FEM one. This procedure is described in details in section 3.

Basing on the obtained estimation one can adaptively refine mesh in three ways: p–, r– or h–refinement. First one means local change of the degree of approximation functions. The disadvantage of this method is that its applicability to nonlinear or three-dimensional problems is uncertain [8]. R–refinement consists in rearrangement of the positions of existing mesh nodes, preserving their total number and approximation degree. Obviously, this technique doesn't guarantee achievement of prespecified accuracy. Also, its implementation is quite difficult. Therefore, the most common practice in adaptive mesh refinement is local resizing of elements of a grid – so called h–adaptivity. Most of the algorithms that use this methodology are rigorously grounded and applicable to different types of problems. However, if we want to keep elements with acceptable errors, we face the problem of preserving mesh conformity, which in general is not trivial and causes distortion of a mesh. Thereby it makes sense to use special methods of solving problems on nonconforming grids. Hereby we use mortar element method [4] since it doesn't impair the convergence rate of the basic finite element method [10].

Therefore, in this paper we present an error estimator, based on the difference in stresses, obtained by FEM and BEM. We use it to construct h–adaptive mesh refinement algorithm that invokes mortar functions for nonconforming finite element grids. Numerical experiments demonstrate usability of the algorithm.

1. The original problem and solving methods. Let us consider an elastic homogeneous isotropic body Ω with boundary Γ. We assume Γ to be divided into Γ_u and Γ_τ so that $\Gamma_u \bigcup \Gamma_\tau = \Gamma$, $\Gamma_u \bigcap \Gamma_\tau = \varnothing$. On Γ_u we have given displacement $u = (u_1, u_2)$, and on Γ_τ – given surface force density $\tau = (\tau_1, \tau_2)$. Let us also divide Ω with some conforming mesh into finite elements Ω_e and denote $\tilde{V}$ to be finite element approximation space. We omit body forces for simplicity. Then the variational formulation of elasticity problem in 2D will be:

Find $u \in \tilde{V}$, which satisfies:

$$a(u,v) = f(v), \ \forall v \in \tilde{V} \tag{1}$$

Here $a(u,v) = \int \sigma(u)\varepsilon(v)d\Omega$ defines deformation energy and $f(v) = \int \tau v d\Gamma_\tau$ defines work of external forces. ε and σ denote strain and stress tensors in the body, respectively, and can be obtained from u as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \quad i, j = 1, 2 \tag{2}$$

$$\sigma_{ij} = \lambda\delta_{ij}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \quad i, j = 1, 2, \tag{3}$$

where δ_{ij} is the Kronecker delta; λ and μ are the Lame parameters.

1.1. Mortar functions. Problem (1) can be solved with standard collocation (taking v as Dirac delta function) or Galerkin method, for example. But, as was mentioned before, to use these methods in adaptive processes, we have to preserve mesh conformability, which causes additional difficulties. The mortar functions [4] allow us to carry out remeshing without conformity. Let us suppose that after remeshing we have Ω divided into subdomains Ω_k with conforming mesh inside every subdomain. Denote Γ_{ij} to be common boundary of Ω_i and Ω_j (so-called interface). We use standard finite element approximation space (let us denote it V_i) inside every subdomain Ω_i, therefore the displacement u, in general, can't be continuous over the interface. Thus impose the weak continuity condition for displacement over every interface Γ_{ij}. Denote t_k, $k = \overline{1,K}$ to be the nodes of mesh of Ω_i (or Ω_j, this choice doesn't affect much [10]), that belong to Γ_{ij}. Using these nodes, let us introduce one-dimensional linear mortar basis ϕ_k, as was shown in [10]:

$$\phi_k(t) = \begin{cases} 0, & t_0 \le t < t_{k-1}; \\ 1, & t_{k-1} \le t < t_k,\ k = 1; \\ \dfrac{t - t_{k-1}}{t_k - t_{k-1}}, & t_{k-1} \le t < t_k,\ 1 < k < K-1; \\ -\dfrac{t - t_{k+1}}{t_{k+1} - t_k}, & t_k \le t < t_{k+1},\ 1 < k < K-1; \\ 1, & t_k \le t < t_{k+1},\ k = K-1; \\ 0, & t_{k+1} \le t < t_K. \end{cases}$$

Fig. 1 illustrates the shapes of the functions.

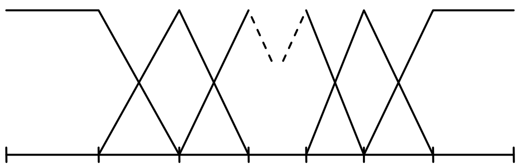


Fig. 1: Mortar basis functions

Then the weak continuity condition will be:

$$\int_{\Gamma_{ij}} \left(u^i(t) - u^j(t)\right)\phi_k(t)\,dt = 0, \quad k = \overline{1,K-1}, \tag{4}$$

where u^i and u^j are the displacements in Ω_i and Ω_j, respectively.

Let us denote $\Lambda = \operatorname{span}\{\phi_k\}$, $V = \prod V_i$ and taking into consideration condition (4), restate the elasticity problem with Lagrange multiplier method:

Find $(u,\lambda) \in (V \times \Lambda)$, which satisfy:

$$\begin{aligned} &a(u,v) + b(\lambda,u) = f(v), \quad \forall v \in V, \\ &b(\mu,u) = 0, \quad \forall \mu \in \Lambda, \end{aligned} \tag{5}$$

where $b(\mu,u) = \int \left(u^i - u^j\right)\mu\, d\Gamma_{ij}$ represents weak continuity over Γ_{ij}. In Ref. [10] it is shown that the error of displacement, obtained from (5), is of rate $O(h^2)$ (when using linear basic functions). Since strain and stress are obtained from displacement derivatives (see (2) and (3)), their errors are of rate $O(h)$.

1.2. Boundary element method. The other way to solve the elasticity problem is the boundary element method. Let us consider its direct formulation.

Denote $G(x,\xi)$ and $F(x,\xi)$ to be fundamental solutions for displacements u and force densities τ respectively. The duality theorem yields the integral equation [2]:

$$u_j(\xi) = \int_\Gamma G_{ij}(x,\xi)\tau_i(x)dx + \int_\Gamma F_{ij}(x,\xi)u_i(x)dx, \quad j = 1,2 \tag{6}$$

Let us divide Γ into N boundary elements Γ_p and pick out m points ξ^{pr}, $r = 1..m$ at each element. Then the boundary integrals in (6) can be divided into N items. Performing this transformation and approaching x to boundary Γ, we have

$$\frac{1}{2}u_j(\xi) = \sum_{p=1}^{N}\int_{\Gamma_p} G_{ij}(x,\xi)\tau_i(x)dx - \sum_{p=1}^{N}\int_{\Gamma_p} F_{ij}(x,\xi)u_i(x)dx \tag{7}$$

Introduce $t \in [-1;1]$ to be a parameter on Γ_p and $N^r(t)$ – Lagrange polynomial basic functions, using nodes ξ^{pr}. Since u and τ in (7) are defined at Γ, we can approximate them with $N^r(t)$:

$$u_i\left(\xi^p(t)\right) = \sum_{r=1}^{m} u_i^{pr}N^r(t), \qquad u_i^{pr} = u_i\left(\xi^{pr}\right), \qquad \xi^p \in \Gamma_p$$

$$\tau_i\left(\xi^p(t)\right) = \sum_{r=1}^{m} \tau_i^{pr}N^r(t), \qquad \tau_i^{pr} = \tau_i\left(\xi^{pr}\right), \qquad \xi^p \in \Gamma_p$$

Applying this approximation to (7) and using the Galerkin method, we find the system

$$\begin{aligned}
&\frac{1}{2}u_j^{qs}\int_{\Gamma_q} N^s(\eta)d\eta = \\
&= \sum_{p=1}^{N}\sum_{r=1}^{m}\tau_i^{pr}\int_{\Gamma_q} N^s(\eta)\int_{\Gamma_p} G_{ij}\left(x(t),\xi(\eta)\right)dtd\eta - \\
&- \sum_{p=1}^{N}\sum_{r=1}^{m}u_i^{pr}\int_{\Gamma_q} N^s(\eta)\int_{\Gamma_p} F_{ij}\left(x(t),\xi(\eta)\right)dtd\eta
\end{aligned}
\qquad
\begin{aligned}
q &= \overline{1,N}, \\
s &= \overline{1,m}, \\
j &= \overline{1,2}
\end{aligned}$$

After solving this system, any displacement or stress inside Ω can be calculated as:

$$u_j(\xi) = \sum_{p=1}^{N}\sum_{r=1}^{m}\left(\tau_i^{pr}\int_{\Gamma_p} G_{ij}\left(x(t),\xi\right)dt - u_i^{pr}\int_{\Gamma_p} F_{ij}\left(x(t),\xi\right)dt\right)$$

$$\sigma_{jk}(\xi) = \sum_{p=1}^{N}\sum_{r=1}^{m}\left(\tau_i^{pr}\int_{\Gamma_p} T_{ijk}\left(x(t),\xi\right)dt - u_i^{pr}\int_{\Gamma_p} E_{ijk}\left(x(t),\xi\right)dt\right)$$

Wendland and Hsiao have proved in [5] that the errors are of rate O(h).

2. Error estimation. Since stresses in FEM and BEM are obtained with different precision, let us build an error estimator using this difference. First denote σ_B and σ_F to be BEM and FEM stresses respectively, σ_T – BEM stresses on a grid, dense enough to consider them to be exact; $\|\Omega\|$ – area of Ω. We want to compare stresses in different points, taking into consideration all the tensor components in equal measure, so introduce effective stress as:

$$\sigma_{eff} = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2}$$

assuming plain strain theory is valid for our problem. Here σ_{33} can be defined as

$$\sigma_{33} = \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11} + \sigma_{22})$$

Then effective stress difference will be:

$$\Delta\sigma_{FB}(x) = \sqrt{\sum_{i,j=1}^{3}(\sigma_{Fij} - \sigma_{Bij})^2}$$

The mesh adaptivity process requires detecting finite elements Ω_e, which need refinement, therefore determine mean root square of effective stress difference for each Ω_e, using L_2–norm over the finite element:

$$\overline{\Delta\sigma_{FB\,\Omega_e}} = \frac{\|\Delta\sigma_{FB}\|_{\Omega_e}}{\|\Omega_e\|} = \frac{\sqrt{\int \Delta\sigma_{FB}{}^2 d\Omega_e}}{\sqrt{\int 1 d\Omega_e}} \tag{8}$$

Likewise, root mean square of effective stress, obtained with BEM, in Ω_e will be:

$$\overline{\sigma_{B\,\Omega_e}} = \frac{\|\sigma_B\|_{\Omega_e}}{\|\Omega_e\|} = \frac{\sqrt{\int \sum_{i,j=1}^{3} \sigma_{Bij}{}^2 d\Omega_e}}{\sqrt{\int 1 d\Omega_e}}$$

Similar formulas can be written for σ_S and σ_T.

Since $\overline{\Delta\sigma_{FB\,\Omega_e}}$ is normalized to the area of the finite element, we can compare this value on different elements; it is an analogue of an absolute error. To find a relative error analogue, we have to normalize (8) to $\overline{\sigma_{B\,\Omega}}$ (but not $\overline{\sigma_{S\,\Omega}}$, as its precision is lower). Note, that we normalize $\overline{\Delta\sigma_{FB\,\Omega_e}}$ to a value, calculated over the whole domain Ω. More common normalization to $\overline{\sigma_{B\,\Omega_e}}$, obtained with integration over a finite element, is not acceptable here, since these values can differ greatly. This leads to “big” relative errors at elements with “small” stresses in adaptivity process it is an undesirable result.

Therefore, we propose to estimate relative error of FEM stresses with

$$\eta = \frac{\overline{\Delta\sigma_{\mathrm{FB}\,\Omega_e}}}{\overline{\sigma_{\mathrm{B}\,\Omega}}} = \frac{\|\Delta\sigma_{\mathrm{FB}}\|_{\Omega_e} / \|\Omega_e\|}{\|\sigma_{\mathrm{B}}\|_{\Omega} / \|\Omega\|} \tag{9}$$

Since this estimator is element-wise, we can use it as an adaptivity criterion.

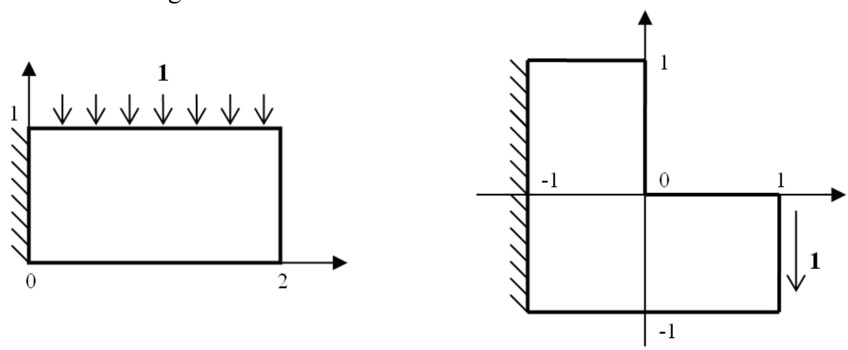
To determine how adequate estimation (9) is, let us also consider an effectivity index of the estimator η [12]:

$$\theta = \frac{\overline{\Delta\sigma_{\mathrm{FB}\,\Omega e}}}{\overline{\sigma_{\mathrm{B}\,\Omega}}} / \frac{\overline{\Delta\sigma_{\mathrm{FT}\,\Omega e}}}{\overline{\sigma_{\mathrm{T}\,\Omega}}}$$

Here $\overline{\Delta\sigma_{\mathrm{FT}\,\Omega_e}}$ is obtained from σ_{S} and σ_{T} similarly to Eq. (8). This effectivity index shows how well $\overline{\Delta\sigma_{\mathrm{FB}\,\Omega_e}}$ approximates real error $\overline{\Delta\sigma_{\mathrm{FT}\,\Omega_e}}$. If θ is close to 1, then the estimation is adequate.

Previous statements hold true only if the diameters of the FEM and BEM grids are the same or close. Through the adaptive mesh refinement, the grids have to be connected. We propose to construct BEM grid as a “trace” of FEM one. In other words, the FEM grid nodes, which belong to Γ, we consider to be BEM nodes. That allows us to compare the results of these methods.

3. Numerical experiments. We have investigated stress error estimator η for problems of plane strain theory, depicted in Fig. 2. Each problem was solved using FEM and BEM. We constructed several conforming and nonconforming meshes for each method to study the applicability of the estimator. As we mentioned before, BEM grid is a “trace” of FEM one. Some of the results are presented in Figures 3–6. In the case of nonconforming mesh we used mortar functions to merge the solution over the interface.



a b

Fig. 2: Geometry and loads of the test problems

The meshes were adapted to the singularities of each problem. Upper and lower left corners in Problem A are the points of change of boundary data type, thus the mesh is finer here. Problem B represents deformation of an L–shaped domain, which has singularity in the inner corner: stress tends to infinity when approaching to this point. In general, the distribution of error estimator recovers these singularities, though using mortar functions worsen the results near the interface (Fig. 4, 6). The reason of this decay becomes obvious when we remember that the numerical solution (displacement) is not continuous over the interface.

Figures 3b) – 6b) represent effectivity of the error estimator η. The effectivity index θ is close to 1 at all the finite elements, except the element straight in the inner corner of Problem B. Therefore, our estimator is adequate.

a)

0.62	0.37	0.24	0.19	0.13	0.09	0.06	0.05
0.30	0.26	0.21	0.15	0.11	0.08	0.06	0.05
0.29	0.25	0.21	0.15	0.11	0.08	0.06	0.05
0.55	0.36	0.24	0.19	0.14	0.09	0.06	0.05

b)

1.10	0.98	0.97	0.98	0.97	0.97	0.97	0.99
0.91	1.00	1.00	1.00	1.00	1.00	0.99	0.98
0.92	1.00	1.00	1.00	1.00	1.00	0.99	0.98
1.07	0.97	0.97	0.97	0.97	0.97	0.97	0.98

Fig. 3: Conforming mesh in Problem A: error estimator η (a), effectivity index θ (b).

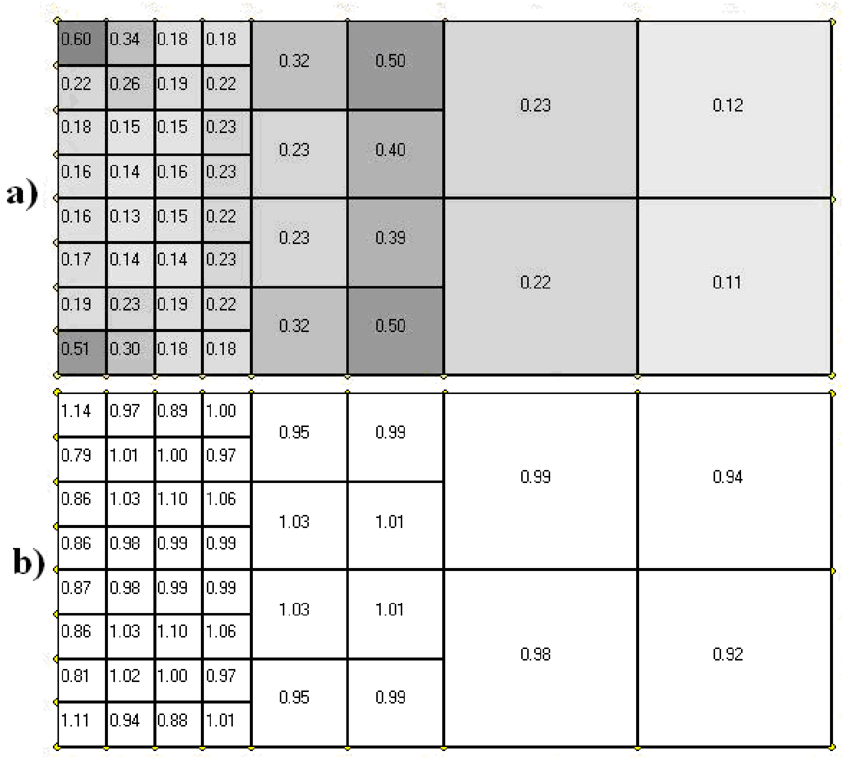


Fig. 4: Nonconforming mesh in Problem A: error estimator η (a), effectivity index θ (b).

4. Conclusions. We have determined the possibility of usage the difference between FEM and BEM for error estimation and mesh adaptivity. We constructed an element-wise estimator based on the difference in stresses, obtained with these methods. The mortar element method was employed to solve the problems with nonconforming FEM grid. The estimator was investigated on several problems and the results indicate its ability to recover singularities. Effectivity indices confirm validity of the estimator, though mortar elements worsen the results near the interfaces. Strict theoretical ground for using this technique on nonconforming grids is a problem of future investigation.

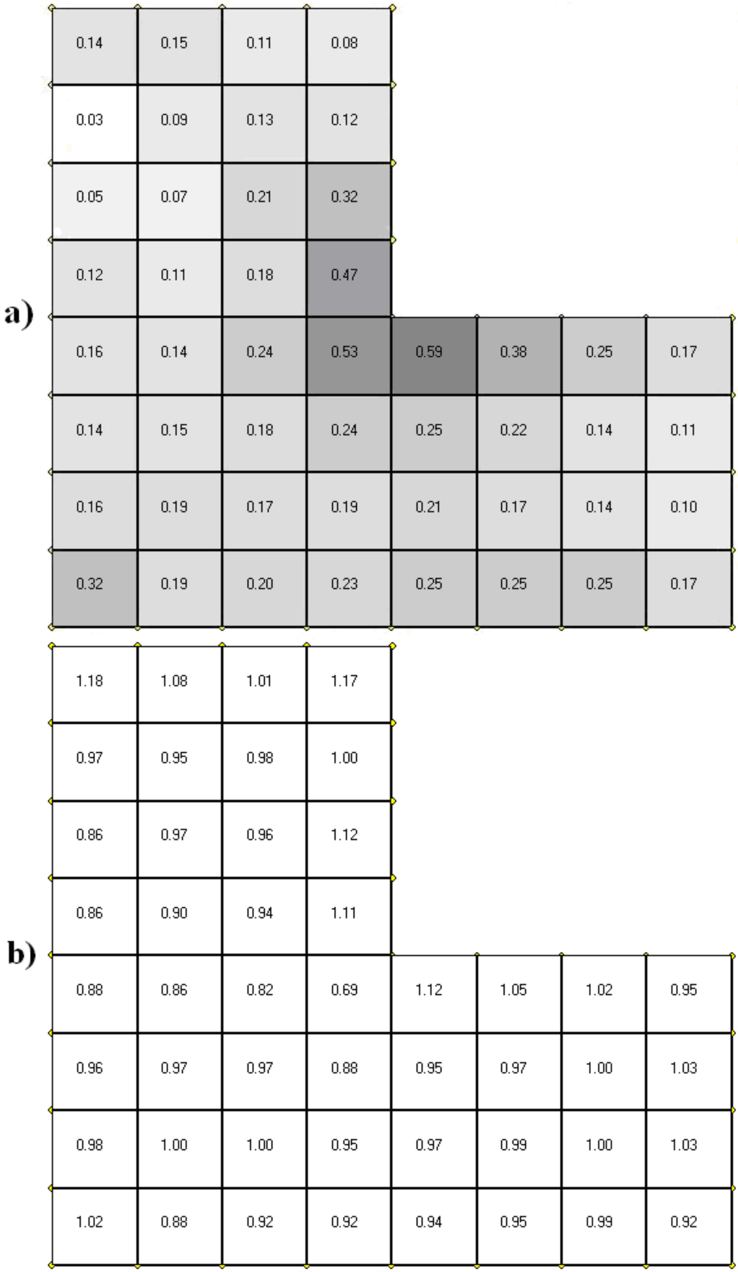


Fig. 5: Conforming mesh in Problem B: error estimator η (a), effectivity index θ (b).

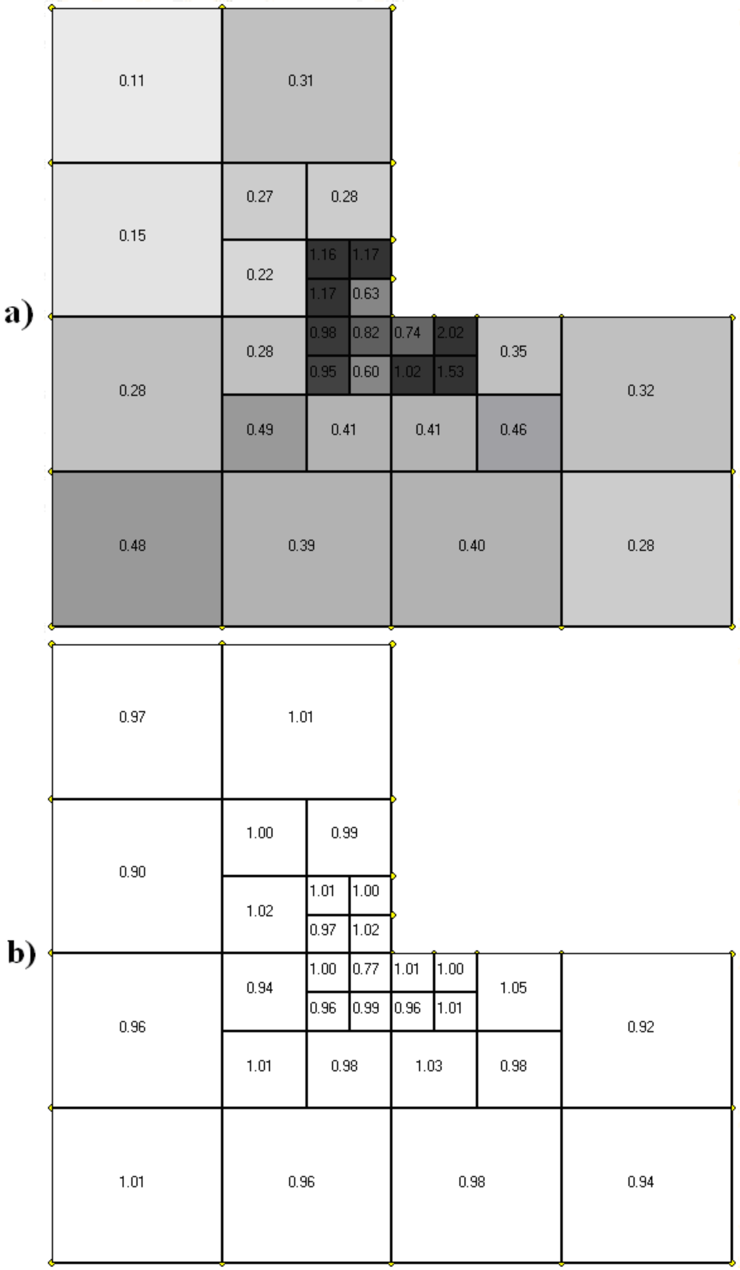


Fig. 6: Nonconforming mesh in Problem B: error estimator η (a), effectivity index θ (b).

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