STATIONARY SOLUTION STABILITY FOR THE PROBLEM OF CREEP IN A ROD SYSTEM

Fedorov V. A.

National Technical University “Kharkov Polytechnic Institute”, Ukraine

The mathematical models of creep in structural elements are essentially nonlinear [1, 2]. As a rule, these equations are solved using numerical methods. The approximate solution can have a significant error. To determine the cause of this, investigating the Lyapunov solution stability for creep equations of investigated structures is critical.

This problem has been investigated extensively in the literature [3–5]. However, the application of the theory to mechanics is related primarily to dynamic processes and motion control. Creep problems were not considered earlier in the literature for solution stability. In this paper, this problem is studied for the stationary solution of a creep problem in a rod system.

1. The creep problem for a rod system. The nonsteady creep of a parallel system of n rods extended by constant force P is shown in Fig. 1.

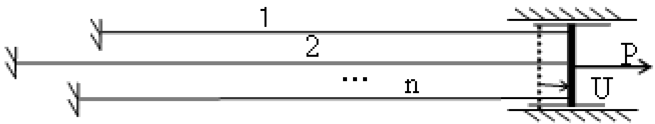

Fig. 1. Rod system

Each rod has individual dimensions (length l_i and cross–section F_i) and mechanical properties (modulus of elasticity E_i and material creep law $\dot{\varepsilon}_i^c = \phi_i(s_i)$, $i = \overline{1,n}$; $s_i = \sigma_i / E_i$ are strains being a dimensionless measure of stresses).

The stability of a stationary solution of the rod system creep problem is investigated. The behaviour of a rod system is described by equilibrium equations, strain compatibility equations and physical relationships for non-stationary creep in matrix form:

$$\hat{\mathbf{r}}^T \hat{\mathbf{s}} = P \qquad (1)$$

$$\hat{\boldsymbol{\varepsilon}} = U \hat{\boldsymbol{\alpha}} \qquad (2)$$

$$\dot{\hat{\boldsymbol{\varepsilon}}} = \dot{\hat{\mathbf{s}}} + \hat{\mathbf{f}}(\hat{\mathbf{s}}) \qquad (3)$$

Here, U is displacement of rod ends (Fig. 1), $\hat{\mathbf{s}}$ is dimensionless stress vector; $\hat{\boldsymbol{\varepsilon}} = (\varepsilon_1 \;\; \varepsilon_2 \; \ldots \varepsilon_n)^T$ is strain vector; $\hat{\mathbf{r}} = (E_1F_1 \;\; E_2F_2 \; \ldots \; E_nF_n)^T$ is specific rod rigidity vector; $\hat{\boldsymbol{\alpha}} = (1/l_1 \;\; 1/l_2 \; \ldots \; 1/l_n)^T$ is rod inverse length vector; and $\hat{\mathbf{f}}(\hat{\mathbf{s}}) = (f_1(s_1) \; f_2(s_2) \ldots f_n(s_n))^T$ is vector of the creep strain rate function.

After excluding strains and displacements from systems (1) to (3), we obtain a linearly–dependent system of common differential equations (SCDE)

$$\dot{\widehat{\mathbf{s}}} = \widehat{\mathbf{g}}(\widehat{\mathbf{s}}), \quad \widehat{\mathbf{g}} = \widehat{\mathbf{r}}^{\mathrm{T}}\widehat{\mathbf{f}}(\widehat{\mathbf{s}})\widehat{\boldsymbol{\alpha}} / \mathrm{R} - \widehat{\mathbf{f}}(\widehat{\mathbf{s}}) \tag{4}$$

$\mathrm{R} = \widehat{\mathbf{r}}^{\mathrm{T}}\widehat{\boldsymbol{\alpha}}$ is rod system elastic rigidity. The linear dependence of velocities of dimensionless stresses follows from equilibrium equation (1) after they have been differentiated with respect to time.

An alternative, linearly independent SCDE is built from system (4), in which the last equation is replaced with equilibrium equation (1) solved for s_n:

$$s_n = (\mathrm{P} - \mathbf{r}^{\mathrm{T}}\mathbf{s}) / r_n$$

and the last equality is substituted into the right-hand parts of the remaining equations:

$$\dot{\mathbf{s}} = \mathbf{g}(\mathbf{s}), \quad \mathbf{g} = \mathbf{r}^{\mathrm{T}}\mathbf{f}(\mathbf{s})\boldsymbol{\alpha} / \mathrm{R} - \mathbf{f}(\mathbf{s}) + r_n f_n[(\mathrm{P} - \mathbf{r}^{\mathrm{T}}\mathbf{s}) / r_n]\boldsymbol{\alpha} / \mathrm{R} \tag{5}$$

From the above-stated it follows that a linearly dependent SCDE (5) is the first integral of the linearly dependent SCDE (4).

Here and further, unless otherwise stated, vectors without parentheses are formed from like vectors in parentheses by deleting the n-th element. Similarly, the square matrices without parentheses will denote like matrices in parentheses, in which the n-th row and the n-th column are deleted.

The initial condition for SCDE (4) and (5) is the elastic solution of the problem.

Using known creep laws [1,2] for stated initial problems, the conditions of the theorem on existence and uniqueness of solution [6] hold.

Let us consider Lyapunov stability for stationary solutions $\widehat{\mathbf{s}}_0$ SCDE (4) $\dot{\widehat{\mathbf{s}}}_0 = \widehat{\mathbf{g}}(\widehat{\mathbf{s}}_0) = \widehat{\mathbf{0}}$ and $\mathbf{s}_0$ SCDE (5) $\dot{\mathbf{s}}_0 = \mathbf{g}(\mathbf{s}_0) = \mathbf{0}$.

2. Stationary solution stability problem. The perturbed stationary solution for linearly dependent system (4) has the form

$$\widehat{\mathbf{s}}(t) = \widehat{\mathbf{s}}_0 + \widehat{\mathbf{v}}(t) \tag{6}$$

where $\widehat{\mathbf{v}}(t)$ is its small perturbation. By substituting perturbed solution (6) in SCDE (4) and assuming continuous differentiability of $\widehat{\mathbf{g}}(\widehat{\mathbf{s}})$ in the neighborhood of $\widehat{\mathbf{s}}_0$, we obtain an autonomous quasilinear system of differential equations $\dot{\widehat{\mathbf{v}}} = \widehat{\mathbf{J}}\,\widehat{\mathbf{v}} + \widehat{\mathbf{O}}\left(\dot{\widehat{\mathbf{v}}}\right)$. Here, $\widehat{\mathbf{J}}$ is Jacobian of the right-hand part of system (4):

$$\widehat{\mathbf{J}} = \widehat{\boldsymbol{A}}\widehat{\mathbf{U}}\widehat{\mathbf{R}}\widehat{\mathbf{F}} / \mathrm{R} - \widehat{\mathbf{F}} \tag{7}$$

where matrices

$$\widehat{\boldsymbol{A}} = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}, \quad \widehat{\mathbf{U}} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad \widehat{\mathbf{R}} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r_n \end{pmatrix},$$

$\widehat{\mathbf{F}} = \widehat{\mathbf{F}}(\widehat{\mathbf{s}}_0)$ is Jacobian of vector $\widehat{\mathbf{f}}$:

$$\widehat{\mathbf{F}} = \begin{pmatrix} \mathrm{df}_1/\mathrm{ds}_1 & 0 & \dots & 0 \\ 0 & \mathrm{df}_2/\mathrm{ds}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathrm{df}_n/\mathrm{ds}_n \end{pmatrix}.$$

According to the theorem on general index stability [3] for twice differentiable functions $\mathrm{f}_i\left(s_i\right)$, the stationary solution for SCDE (4) is asymptotically stable if the general index of SCDE in variations of SCDE (7) with respect to stationary solution $\widehat{\mathbf{s}}_0$

$$\dot{\widehat{\mathbf{v}}} = \widehat{\mathbf{J}}\widehat{\mathbf{v}} \tag{8}$$

is negative. Since the SCDE (8) is stationary and linear, its general index is negative if the real parts of eigenvalues $\widehat{\lambda}_i$ of Jacobian $\widehat{\mathbf{J}}$ are negative [3]: $\mathrm{Re}(\widehat{\lambda}_i) < 0, \quad i = \overline{1,n}.$

Similar reasoning with respect to a linearly independent SCDE (5) yields a SCDE in variations for perturbation vector $\mathbf{v}(t)$:

$$\dot{\mathbf{v}} = \mathbf{J}^*\mathbf{v}\,, \tag{9}$$

where the Jacobian of the right-hand part of SCDE (5) has the form

$$\mathbf{J}^* = \mathbf{J} + \mathbf{Y} \tag{10}$$

Here

$$\mathbf{J} = A\mathbf{URF}/R - \mathbf{F} \tag{11}$$

$$\mathbf{Y} = -A\bar{\mathbf{U}}\mathbf{R}(\mathrm{df}_n / \mathrm{d}s_n)/\mathrm{R} \tag{12}$$

Equation (1) yields the relationship

$$\mathrm{v}_n = -\mathbf{r}^{\mathrm{T}}\mathbf{v}/\mathrm{r}_n \tag{13}$$

The condition of asymptotic stability of a stationary solution of a linearly independent SCDE (5) has the form

$$\mathrm{Re}(\lambda_i) < 0, \quad i = \overline{1, n-1} \tag{14}$$

where λ_i are eigenvalues of matrix $\mathbf{J}^*$.

Note that the SCDE in variations (8) is also linearly independent as the initial SCDE (4) is. The SCDE in variations (9), being a first integer of SCDE (8), is linearly independent as the initial SCDE (5) is.

3. Stability of stationary solution. The objective of the study was investigating the properties of a linearly independent SCDE (5). However, the linearly independent SCDE (4) and its SCDE in variations (8), due to their simpler structure, are auxiliary ones. Assume that the creep stability of rod materials, i.e. their creep laws are increasing stress functions:

$$\mathrm{df}_i / \mathrm{ds}_i > 0; \quad i = \overline{1, n} \tag{15}$$

Lemma 1. *If the creep strain rates are increasing stress functions (26), then the eigenvalues of Jacobian* $\widehat{\mathbf{J}}$ *(7) are real and its fixed sign property matches the fixed sign property of matrix*

$$\widehat{\mathbf{B}} = \widehat{\mathbf{U}} - \mathrm{R}\widehat{A}^{-1}\widehat{\mathbf{R}}^{-1} \tag{16}$$

Since diagonal matrices are commutative, Jacobian $\widehat{\mathbf{J}}$ is written as follows

$$\widehat{\mathbf{J}} = \widehat{A}\widehat{\mathbf{B}}\widehat{\mathbf{R}}\widehat{\mathbf{F}} / \mathrm{R}. \tag{17}$$

Since all diagonal matrices in (17) are positively determined, they can be presented as $\mathbf{D} = \mathbf{D}^{1/2}\mathbf{D}^{1/2}$ [7]. Since matrices $\mathbf{DC}$ and $\mathbf{D}^{1/2}\mathbf{C}\mathbf{D}^{1/2}$ are similar ($\mathbf{C}$ is an arbitrary matrix), it is easily seen that matrix

$$\widehat{\mathbf{J}}_{\mathrm{s}} = \widehat{\mathbf{D}}^{1/2}\widehat{\mathbf{B}}\widehat{\mathbf{D}}^{1/2} \tag{18}$$

is similar to matrix $\widehat{\mathbf{J}}$ (7). Here, $\widehat{\mathrm{D}} = \widehat{A}\widehat{\mathbf{R}}\widehat{\mathbf{F}} / \mathrm{R}$ is a positively defined diagonal matrix, and matrix $\widehat{\mathbf{B}}$ (16) is symmetrical.

The eigenvalues of symmetrical matrix $\widehat{\mathbf{J}}_{\mathrm{s}}$ (18) and matrix $\widehat{\mathbf{J}}$ similar thereto are real [7], which is what had to be proved.

Further, we will show that the properties of fixed-sign properties of matrices $\widehat{\mathbf{J}}_{\mathrm{s}}$ and $\widehat{\mathbf{B}}$ match. If $\widehat{\mathbf{B}}$ is nonpositively defined ($\widehat{\mathbf{x}}^{\mathrm{T}}\widehat{\mathbf{B}}\widehat{\mathbf{x}} \le 0$), then, by replacing vector $\widehat{\mathbf{x}}$ with $\widehat{\mathbf{x}} = \widehat{\mathbf{D}}^{1/2}\widehat{\mathbf{y}}$, we obtain that $\widehat{\mathbf{J}}_{\mathrm{s}}$ is nonpositively defined: $\widehat{\mathbf{y}}^{\mathrm{T}}(\widehat{\mathbf{D}}^{1/2}\widehat{\mathbf{B}}\widehat{\mathbf{D}}^{1/2})\widehat{\mathbf{y}} = \widehat{\mathbf{y}}^{\mathrm{T}}\widehat{\mathbf{J}}_{\mathrm{s}}\widehat{\mathbf{y}} \le 0$. This proves Lemma 1.

Let us study the properties of matrix

$$\widehat{\mathbf{B}} = \begin{pmatrix} e_1 & 1 & \ldots & 1 \\ 1 & e_2 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & e_{\mathrm{n}} \end{pmatrix}, \tag{19}$$

where

$$\mathrm{e_i} = 1 - \left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \eta_{\mathrm{j}} \right) / \eta_{\mathrm{i}} \tag{20}$$

$$\eta_{\mathrm{i}} = \mathrm{r_i}\alpha_{\mathrm{i}}. \tag{21}$$

Lemma 2. *The leading minors of matrix* $\widehat{\mathbf{B}}$ *(19) of the* k*-th order have the following form:*

$$\mathrm{M_k} = \prod_{\mathrm{i}=1}^{\mathrm{k}} (\mathrm{e_1} - 1) + \sum_{\mathrm{j}=1}^{\mathrm{k}} \prod_{\mathrm{i}=1\, \mathrm{i}\ne \mathrm{j}}^{\mathrm{k}} (\mathrm{e_i} - 1). \tag{22}$$

It is easily verified that for k=1, 2, 3, formula (22) is confirmed.

Now, according to the mathematical induction method, let us confirm the formula by applying it to minors of the k-th and (k-1)-th orders in the expansion of minor $\mathrm{M_k}$ for elements of its k-th column:

$$M_k = M_{k-1}e_k - \sum_{e=1}^{k-1} M_{k-1}^{(1)}, \tag{23}$$

where $M_{k-1}^{(1)}$ is (k-1)-th order determinant obtained from minor M_{k-1} by substituting a unit for element e_l. By applying formula (33) to the determinants in the right-hand side of equality (34), we obtain

$$\begin{aligned} M_k = &\left[\prod_{i=1}^{k-1}(e_l - 1) + \sum_{j=1}^{k-1}\prod_{i=1\, i\neq j}^{k-1}(e_i - 1)\right]e_k - \\ &-\sum_{l=1}^{k-1}\left[\prod_{i=1}^{k-1}(e_i^{(1)} - 1) + \sum_{j=1}^{k-1}\prod_{i=1\, i\neq j}^{k-1}(e_i^{(1)} - 1)\right]. \end{aligned} \tag{24}$$

Here $e_i^{(1)} = e_i\left(1-\delta_{il}\right)+\delta_{il}$, where δ_{il} is Kronecker's symbol. If we take into account that $\prod_{i=1}^{k-1}\left(e_i^{(1)}-1\right)=0$, and $\sum_{j=1}^{k-1}\prod_{i=1;\, i\neq j}^{k-1}\left(e_i^{(1)}-1\right)=\prod_{i=1;\, i\neq j}^{k-1}\left(e_i-1\right)$, the right-hand part of formula (35) is easy to transform to (34), which is what had to be proved in Lemma 2.

Lemma 3. *Matrix* $\widehat{\mathbf{B}}$ *is defined nonpositively.*

Further, it is convenient to investigate the properties of matrix $-\widehat{\mathbf{B}}$, whose leading minors are

$$\overline{M}_k = \left(-1\right)^k M_k. \tag{25}$$

Let us prove that the leading minors of matrix $-\widehat{\mathbf{B}}$ are non-negatively defined. For proving, we will apply $\eta_i > 0;\ i = \overline{1,n}$ (21). Then

$$\sum_{j=1}^{k}\eta_j \Big/ \sum_{j=1}^{n}\eta_j < 1 \quad (k = \overline{1, n-1}). \tag{26}$$

After substituting $\eta_i / \sum_{j=1}^{n}\eta_j = \left(1-e_i\right)^{-1}$ from (20) into relationship (26), the latter is converted to the form

$$\frac{\prod_{i=1}^{k}\left(e_i-1\right)+\sum_{j=1}^{k}\prod_{i=1;\, i\neq j}^{k}\left(e_i-1\right)}{\prod_{i=1}^{k}\left(e_i-1\right)} > 0 \quad (k = \overline{1, n-1}). \tag{27}$$

With account of equalities (22) and (25) as well as that $e_i - 1 < 0$ (20), inequality (27) is rearranged in the form

$$\overline{M}_k > 0 \quad (k = \overline{1, n-1}) \tag{28}$$

For k = n inequality (26) and hence (28) are transformed to equality:

$$\overline{M}_n = \det(-\widehat{\mathbf{B}}) = 0 \tag{29}$$

Since any transpositions of rods numbering has no affect on the above results, it is obvious that inequalities (28) are true not only for all leading but also for all principal minors of matrix $-\widehat{\mathbf{B}}$. This, with account of equality (29), is the necessary and sufficient condition for non-negative definiteness of matrix $-\widehat{\mathbf{B}}$ [7], and hence, for non-positive definiteness of matrix $\widehat{\mathbf{B}}$. This proves Lemma 3.

Theorem 1. *If the creep strain rates are increasing stress functions (15), then the stationary solution of the linearly independent system (5) is asymptotically stable.*

The nonpositive definiteness of matrix $\widehat{\mathbf{B}}$ as well as that its rank is a unit less than its order, implies the following for its eigenvalues:

$$\widehat{\mu}_i < 0; \quad (i = \overline{1, n-1}); \quad \widehat{\mu}_n = 0$$

By virtue of an identical fixed sign property of matrix $\widehat{\mathbf{J}}_s$ with that of $\widehat{\mathrm{B}}$, their symmetry and rank equality, as well as similarity of matrices $\widehat{\mathbf{J}}_s$ and $\widehat{\mathbf{J}}$, for matrix $\widehat{\mathbf{J}}$ eigenvalues we have

$$\widehat{\lambda}_i < 0; \quad (i = \overline{1, n-1}); \quad \widehat{\lambda}_n = 0 \tag{30}$$

Since the linearly independent SCDE (5), as well as its SCDE in variations (9) are the first integrals of respective linearly independent systems, the set of Jacobian $\mathbf{J}$ eigenvalues is a subset of the set of Jacobian $\widehat{\mathbf{J}}$ eigenvalues (30):

$$\lambda_i = \widehat{\lambda}_i < 0; \quad (i = \overline{1, n-1}) \tag{31}$$

The null eigenvalue in this set is absent because it is a consequence of the linear dependence of the SCDE (8), and SCDE (9) is linearly independent. The result obtained (31) meets the condition of asymptotic stability (14) of the stationary solution of linearly independent SCDE (5). This proves Theorem 1.

Theorem 2. *The sufficient conditions (15) of the asymptotic stability of the stationary solution of linearly independent system (5) can be weakened:*

$$df_i / ds_i > 0 \quad \left(i = \overline{1, n-1}\right) \tag{32}$$

$$df_n / ds_n \geq 0 . \tag{33}$$

Indeed, if $df_n / ds_n = 0$, then matrix $\mathbf{Y} = \mathbf{0}$, Jacobian $\mathbf{J}^* = \mathbf{J}$, and hence, it is formed from Jacobian $\widehat{\mathbf{J}}$ by deleting the *n*-th row from the *n*-th column. Then it is easily seen that the negativity of Jacobian $\mathbf{J}$ eigenvalues is proved using the same algorithm as for proving the nonpositiveness of Jacobian $\widehat{\mathbf{J}}$ eigenvalues. This means that the stationary solution for linearly independent system (5) is asymptotically stable (5), which is what had to be proved.

4. Conclusions. The paper is the first attempt in investigating the conditions of the asymptotic stability of solutions for structure creep equations. The theoretical results obtained for a rod system allow formulating the following practical recommendations.

To obtain asymptotically stable stationary solutions, a linearly independent SCDE (5) should be used, whereas the defining relationships used should be those that meet necessary asymptotic stability conditions (32) and (33).

If the system has more than one elastic rod, then for the rods $df_i / ds_i = 0$, and conditions (32) and (33) are not satisfied. To ensure asymptotic stability of a stationary solution, these rods should be reduced to one fictitious rod with equivalent stiffness. Then, the conditions of theorem 2 will be satisfied.

If the system includes an elastic rod, and the creep of all other ones is described by Norton´s law $\dot{s}_i = A_i s_i^{m_i}$ $(i = \overline{1, n-1})$, then the stationary solution $s_i = 0$ $(i = \overline{1, n-1})$ fails to satisfy asymptotic stability conditions (32) and (33). Under such circumstances, it is preferable to use Prandtl´s law $\dot{s}_i = C_i \sinh(k_i s_i)$ $(i = \overline{1, n-1})$, for which the stated stationary solution satisfies asymptotic stability conditions.

The further development of this problem lies in a direction of research into stability of nonstationary solution to the equations of creep in a rod system. Results presented in the paper can also be useful for modelling of longitudinal creep in fibrous composites and for the research into stiffness of the corresponding equations [8].

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