FORMAL POWER SERIES AND IMPLICIT LINEAR DIFFERENTIAL EQUATION IN A BANACH SPACE

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Let E be a Banach space and A be a closed linear operator on E with a domain of definition $D(A)$. We do not suggest that $D(A)$ is dense in E. The meaningful example of such an operator can be an operator $A = d^2/dx^2$ on the space $C[0,1]$ with the domain of definition $D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$.

We study entire solutions of the inhomogeneous linear differential equation

$$w' = Aw + f(z). \quad (1)$$

We suppose that $f : \mathbb{C} \to E$ is an E–valued entire function and under a solution of this equation we understand a holomorphic in a neighborhood of zero E–valued function $w(z)$, such that $w(z) \in D(A)$ and Equation (1) is fulfilled in the same neighborhood.

First we consider a purely algebraic situation, namely consider the Cauchy problem for the inhomogeneous equation

$$\begin{cases} w' = Aw + f \\ w(0) = w_0 \end{cases} \quad (2)$$

in the space of formal power series. Now here E is an arbitrary vector space, $A : E \to E$ is a linear operator and $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a formal power series with coefficients from E. Than using the method of unknown coefficients it can be shown that the Cauchy problem (2) has the unique solution in the form of formal power series [1]. Note that the simple explicit formulas for this solution can not be obtained. The corresponding recursion relations can be founded only. If E is a Banach space and A is a linear operator with nontrivial domain of definition than the situation essentially thickens, because these evaluations are incorrect. If A is bounded and f is an entire function then the Cauchy problem (2) has always the unique entire solution

$$w(z) = e^{zA} w_0 + \int_0^z e^{-(z-\zeta)A} f(\zeta)\, d\zeta. \quad (3)$$

Usually in applications the operator A of Problem (2) is unbounded and in this case situation is more complicated. One of the reasons is impossibility to determine an exponent of unbounded operator in general case. In the theory of operator semigroups they consider a special class of operators: generating operators of semigroups and their generalizations. Under the corresponding restrictions on the right-hand member we can get a solution of the Cauchy problem for inhomogeneous equation on the semi-axis

$$\begin{cases} u' = Aw + f(t), \quad t > 0 \\ \qquad u(0) = u_0 \end{cases} \tag{4}$$

of the following form

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s)\, ds, \tag{5}$$

where $\{T(t)\}_{t>0}$ is the corresponding semigroup. Note that for finding a solution of the inhomogeneous equation the homogeneous equation solutions of the Cauchy problem should be known. There are so many works at this theory. It is well-studied theory (see, for example, [2]–[5]). The equation on the semi-axis with an operator which has a nondense domain was studied in works of P. Sobolevsky, Ju. Silchenko, Da Prato, and E. Sinestrati ([6]–[8]).

The properties of holomorphic and entire solutions of Equation (1) for the case when the operator A is unbounded were also studied in numerous works (see, for example, [3], [9]–[12]).

In the present paper the entire solutions of the Equation (1) were studied using analysis of the *implicit* inhomogeneous differential equation

$$Tw' + g(z) = w, \tag{6}$$

where T is a bounded linear operator on E. In addition for studying the properties of entire solutions of Equation (6) is used explicit formula

$$w(z) = \sum_{n=0}^{\infty} T^n g^{(n)}(z). \tag{7}$$

We prove that if the pair (T, g) is small in that sense, that $\rho(T)\sigma(g) < 1$, where $\rho(T)$ is the spectrum radius of T and $\sigma(g)$ is the exponential type of g, then Equation (6) has the unique entire solution of exponential type $\sigma(g)$ and also this solution continuously depends on g in the corresponding topology (see Theorem 2 and Corollary 5). Besides, we obtain an integral representation of the Cauchy type for Solution (7) of Equation (6) (see Theorem 7). In this case we should consider the *formal* Laurent series

$$\mathcal{E}_T(\zeta) = \sum_{n=0}^{\infty} \frac{n!T^n}{\zeta^{n+1}} \quad .$$

and *formal* integral in the space of formal Laurent series (see [13], [14]). From Theorem 2 we obtain Theorem 4 on well-posedness of Equation (1) in a space of entire functions of exponential type. Having obtained results are illustrated the examples which relate to the theory ODE and PDE.

On the occasion of implicit linear differential equations in a Banach space we refer to [15]–[17] and to bibliographies were there.

Holomorphic solutions in the form (7), may be at first, were studied in [19], where the case of $g(z)$ being an entire function of zero exponential type is considered. At the paper [18] for studying the behavior of entire solutions of Equation (6) was applied the other technique. In the works [20], [21] holomorphic solutions of linear differential equations in a Banach space over a non-Archimedean field are studied.

Main results. Let E be a Banach space and $\sigma > 0$.

Consider the set E_σ of all entire E–valued functions $f(z)$ for which

$$\sup_{z\in\mathbb{C}} \|f(z)\| e^{-\sigma|z|} < +\infty .$$

Then E_σ is the Banach space with respect to the norm $\|f\|_\sigma = \sup_{z\in\mathbb{C}} \|f(z)\| e^{-\sigma|z|}$.

For $0 < \sigma \le \infty$ we set $\tilde{E}_\sigma = \bigcup_{\sigma_1<\sigma} E_{\sigma_1}$. Then $\tilde{E}_\sigma$ is the space of entire $E-$valued function of exponential type, that is less than σ (if $\sigma = \infty$, then $\tilde{E}_\infty$ is the space of arbitrary functions of exponential type). We shall consider this space with the natural topology of inductive limit of Banach spaces.

Lemma 1. *Let* $f \in E_\sigma$. *Then* $f^{(n)} \in E_\sigma$ *for all* $n \in \mathbb{N}$ *and* $\left\|f^{(n)}(z)\right\| \le \sqrt{2\pi n} e^{\frac{1}{12n}} \sigma^n e^{\sigma|z|} \|f\|_\sigma$, *that is* $\left\|f^{(n)}\right\|_\sigma \le \sqrt{2\pi n} e^{\frac{1}{12n}} \sigma^n \|f\|_\sigma$. *Thus the operator of differentiation* D *is bounded on* E_σ *and* $\left\|D^n\right\|_\sigma \le \sqrt{2\pi n} e^{\frac{1}{12n}} \sigma^n$.

Proof. Let $r > 0$. According to the Cauchy integral formula $f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|\zeta - z| = r} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$. From here

$$\left\|f^{(n)}(z)\right\| \le \frac{n!}{2\pi} \cdot 2\pi r \max_{|\zeta - z| = r} \left(\|f(\zeta)\| e^{-\sigma|\zeta|} \right) \cdot \frac{e^{\sigma|\zeta|}}{|\zeta - z|^{n+1}} \le \frac{n!}{r^n} \|f\|_\sigma \max_{|\zeta - z| = r} e^{\sigma|\zeta|} =$$

$= \frac{n!}{r^n} \|f\|_\sigma \max_{|\zeta - z| = r} e^{\sigma|z + (\zeta - z)|} \le \frac{n!}{r^n} \|f\|_\sigma e^{\sigma|z|} e^{\sigma r}$. Minimizing the left part of this inequality with $r > 0$, we have $\left\|f^{(n)}(z)\right\| \le \frac{n! e^n}{n^n} \sigma^n e^{\sigma|z|} \|f\|_\sigma$. As $n! \le \sqrt{2\pi n} e^{\frac{1}{12n}} \left(\frac{n}{e}\right)^n$ then $\left\|f^{(n)}(z)\right\| \le \sqrt{2\pi n} e^{\frac{1}{12n}} \sigma^n e^{\sigma|z|} \|f\|_\sigma$. From here we have $\sup_{z\in\mathbb{C}} \left\|f^{(n)}(z)\right\| e^{-\sigma|z|} \le \sqrt{2\pi n} e^{\frac{1}{12n}} \sigma^n \|f\|_\sigma$.

Let $T : E \to E$ be an arbitrary bounded linear operator. We shall denote the spectral radius of T as $\rho(T)$.

Theorem 2. *Let* $\sigma_0 = \dfrac{1}{\rho(T)}$, $\sigma < \sigma_0$ *and* $g(z)$ *is an entire function of exponential type* σ. *If* $\rho(T) = 0$, *i.e.* T *is quasinilpotent, then we suppose that* $\dfrac{1}{\rho(T)} = +\infty$. *Then Equation (1) has a unique entire solution of exponential type* σ, $w(z) = \sum_{n=0}^{\infty} T^n g^{(n)}(z)$, *and this solution continuously depends on* g *in the topology of the space* $\tilde{E}_{\sigma_0}$.

Proof. Let $g \in E_{\sigma_1}$, where $\sigma < \sigma_1 < \sigma_0$. Consider the series $\sum_{n=0}^{\infty} T^n g^{(n)}(z)$ and show that this series is uniformly convergent in any disk. Let $|z| \le R$. Then according to Lemma 1 we obtain for $n \ge 1$: $\left\|T^n g^{(n)}(z)\right\| \le \left\|T^n\right\| \left\|g^{(n)}\right\| \le$ $\le \left\|T^n\right\| \sqrt{2\pi n} \cdot e^{\frac{1}{12n}} \sigma_1^n e^{\sigma_1 |z|} \|g\|_{\sigma_1} \le e^{\sigma_1 R} \|g\|_{\sigma_1} \left\|T^n\right\| \sqrt{2\pi n} \cdot e^{\frac{1}{12n}} \sigma_1^n$, $|z| \le R$ and the series $\sum_{n=0}^{\infty} \left\|T^n\right\| \sqrt{2\pi n} \cdot e^{\frac{1}{12n}} \sigma_1^n$ converges, as according to the Gelfand formula $\sqrt[n]{\left\|T^n\right\| \sigma_1^n} \to \rho(T)\sigma_1 < 1$. So the function $w(z) = \sum_{n=0}^{\infty} T^n g^{(n)}(z)$ is entire and it is easy to check that $w(z)$ is the solution of Equation (6). Besides that

$$\|w(z)\| \le \sum_{n=0}^{\infty} \left\|T^n g^{(n)}(z)\right\| \le \|g(z)\| + \left(\sum_{n=1}^{\infty} \sqrt{2\pi n} \cdot e^{\frac{1}{12n}} \sigma_1^n \left\|T^n\right\| \right) \|g\|_{\sigma_1} e^{\sigma_1 |z|}. \quad (8)$$

Hence $w \in E_{\sigma_1}$. Therefore $w(z)$ has exponential type which is not great than σ. As $g(z) = w(z) - Tw'(z)$ then an exponential type of $w(z)$ can not be less σ. Thus, it equals σ. Proof that $w(z)$ is a unique solution of Equation (6), having exponential type that is less than σ. Let $w(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire solution of exponential type that is less than σ_0 for the homogeneous equation $Tw' = w$. Then one can easy show that $c_0 = n! T^n c_n$ (see Lemma 2.1 [18]). Therefore $\sqrt[n]{\|c_0\|} \le \sqrt[n]{n! \|c_n\|} \cdot \sqrt[n]{\left\|T^n\right\|}$. Hence $\lim_{n\to\infty} \sqrt[n]{\|c_0\|} \le \sigma_0 \cdot \rho(T) < 1$, that is $\lim_{n\to\infty} \sqrt[n]{\|c_0\|} = 0$ and $c_0 = 0$. Note that the function $w^{(k)}(z)$ satisfies the homogeneous equation $Tw' = w$ and it is an entire function of exponential type σ. Therefore $c_k = 0$, $k \in \mathbb{N}$, that is $w = 0$. Thus, the uniqueness is proved. The continuous dependence w from g in the topology of $\tilde{E}_{\sigma_0}$ follows from Inequality (8). Theorem 2 is completely proved.

Show that for the case of entire function $g(z)$ being not of exponential type even if $\rho(T) = 0$ Equation (6) can have no smooth solution on $[0, t_0]$, $t_0 > 0$ at all.

Example 3. Let E be a Hilbert space with an orthonormalized basis $\{e_n\}_{n=0}^{\infty}$, T be the weighted shift operator such that $Te_n = \frac{1}{\sqrt{n+1}} e_{n+1}$, and $g(z) = e^{z^2} e_0$. If $w(t) = \sum_{n=0}^{\infty} w_n(t) e_n$ is a solution of Equation (6) on the real axes then

$$\begin{cases} e^{t^2} = w_0(t) \\ \frac{1}{\sqrt{n+1}} w'(t) = w_{n+1}(t), \quad n \geq 0. \end{cases}$$

Hence, $w_n(t) = \frac{1}{\sqrt{n!}} \left(e^{t^2}\right)^{(n)}$ and $w_{2n}(0) = \frac{\sqrt{(2n)!}}{n!}$. Therefore $\sum_{n=0}^{\infty} |w_n(0)|^2 = +\infty$, it contradicts to that w is E–valued function.

Let us now consider the linear differential Equation (1), where A is a closed operator on E with domain of definition $D(A)$, that may be not dense in E.

Theorem 4. *Let the operator* A *have a bounded inverse one and* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be an entire function of exponential type, that is less than* $\sigma_0 = \frac{1}{\rho(A^{-1})}$. *Then Equation (1) has a unique entire solution of exponential type, that is less* σ_0, *as* $w(z) = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}(z)$, *and this solution continuously depends on* f *in the topology of the space* $\tilde{E}_{\sigma_0}$. *Namely if* $0 < \sigma_1 < \sigma_0$, $f \in E_{\sigma_1}$ *and* $w = Sf$, *where* $Sf = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}$, *then* S *is bounded on* E_{σ_1} *and* $\|S\| \leq \|A^{-1}\| \left(1 + \sqrt{2\pi} \sum_{n=1}^{\infty} \sqrt{n} e^{\frac{1}{12n}} \|A^{-n}\| \sigma_1^n\right)$. *Moreover the*

Cauchy problem

$$\begin{cases} w' = Aw + f \\ w(0) = w_0 \end{cases} \tag{9}$$

has an entire solution of exponential type, that is less σ_0*, if and only if* $w_0 + \sum_{n=0}^{\infty} n! A^{-(n+1)} c_n = 0$.

Proof. Let $T = A^{-1}$ and $g(z) = -A^{-1} f(z)$. If $\sigma_1 < \sigma_0$ and $f \in E_{\sigma_1}$, then as operator A^{-1} is bounded. Hence $g \in E_{\sigma_1}$. Note that Equation (1) is equivalent to Equation (6). According to Theorem 2 Equation (1) has the unique entire solution $w(z) = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}(z)$. At that from Inequality (8) it follows

$$\|w(z)\| \le \|A^{-1} f(z)\| + \sqrt{2\pi} \sum_{n=1}^{\infty} \sqrt{n} e^{\frac{1}{12n}} \sigma_1^n \|A^{-n}\| \cdot \|A^{-1}\| \|f\|_{\sigma_1} \cdot e^{\sigma_1 |z|} \le$$

$$\le \|A^{-1}\| \left(\|f(z)\| + \sqrt{2\pi} \sum_{n=1}^{\infty} \sqrt{n}\, e^{\frac{1}{12n}} \sigma_1^n \|A^{-n}\| \cdot \|f\|_{\sigma_1} \cdot e^{\sigma_1 |z|} \right).$$

Therefore $w \in E_{\sigma_1}$ and $\|w\|_{\sigma_1} \le \|A^{-1}\| \left(1 + \sqrt{2\pi} \sum_{n=1}^{\infty} \sqrt{n} e^{\frac{1}{12n}} \|A^{-n}\| \sigma_1^n \right) \cdot \|f\|_{\sigma_1}$,

that is S is bounded on E_{σ_1} and $\|S\| \le \|A^{-1}\| \left(1 + \sqrt{2\pi} \sum_{n=1}^{\infty} \sqrt{n} e^{\frac{1}{12n}} \|A^{-n}\| \sigma_1^n \right)$. In conclusion of proof it is enough to note that $w(0) = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}(0) = -\sum_{n=0}^{\infty} n! A^{-(n+1)} c_n$.

Consider now the space $\tilde{E}_\infty = \bigcup_{\sigma > 0} E_\sigma$ of all E–valued entire functions of exponential type with the natural topology of inductive limit of Banach spaces.

Corollary 5. *Let* T *be a bounded quasinilpotent operator (i.e.* $\sigma(T) = \{0\}$*) and* $g(z) = \sum_{n=0}^{\infty} c_n z^n$ *be an arbitrary* E*–valued entire function of exponential type. Then Equation (6) has a unique entire solution of exponential type* $w(z) = \sum_{n=0}^{\infty} T^n g^{(n)}(z)$*, and this solution continuously depends of* g *in the topology of the space* $\tilde{E}_\infty$.

Proof. Both an existence of the solution as $w(z) = \sum_{n=0}^{\infty} T^n g^{(n)}(z)$ and its uniqueness were proved in Theorem 2. Inequality (8) show that if $g \in E_\sigma$, then $w \in E_\sigma$, and $\|w\|_\sigma \le \|g\|_\sigma + \left(\sum_{n=1}^{\infty}\sqrt{2\pi n}\, e^{\frac{1}{12n}}\sigma^n \left\|T^n\right\|\right)\|g\|_\sigma$. Hence the operator S given as $Sg = \sum_{n=0}^{\infty} T^n g^{(n)}$, is continuous in every space E_σ. The statement is proved.

Remark 6. Both the existence and the uniqueness of the solution of Equation (6) in the space of entire functions of exponential type were proved by other technique (which was more complicated) in Theorem 2.6 [18].

Now we obtain the Cauchy type integral representation from Theorem 2 for Equation (6). For this we need to introduce additional definitions. Let V be a vector space. Let us denote by $V\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$ the space of all formal Laurent series with coefficients from V. For $g(\zeta) = \sum_{n=-\infty}^{\infty} b_n \zeta^n \in V\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$ set

$$\oint g(\zeta)\, d\zeta \overset{\text{def}}{=} 2\pi i\, b_{-1}. \tag{10}$$

We call the linear map $V\left[\left[\zeta,\frac{1}{\zeta}\right]\right] \to V$ given by (10) the Cauchy–Laurent integral.

For bounded linear operator $T: E \to E$ denote by $\mathcal{E}_T(\zeta)$ the formal Laplace–Borel' transformation of Fredholm resolvent of T:

$$\mathcal{E}_T(\zeta) = \sum_{n=0}^{\infty} \frac{n!T^n}{\zeta^{n+1}}.$$

Note that $\mathcal{E}_T(\zeta) \in B(E)\left[\left[\frac{1}{\zeta}\right]\right]$, where $B(E)$ is the space of all bounded linear operators on the Banach space E. We define the expression $\mathcal{E}_T(\zeta - z)$ as the following element of the space $B(E)[[z]]\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$:

$$\mathcal{E}_T(\zeta - z) = \sum_{n=0}^{\infty} \frac{n!T^n}{(\zeta - z)^{n+1}} \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{\zeta^{n+1}} \frac{n!T^n}{\left(1 - \frac{z}{\zeta}\right)^{n+1}} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{z^k}{\zeta^k}\right)^{n+1} \frac{n!T^n}{\zeta^{n+1}}.$$

Theorem 7. *Under the terms and notations of Theorem 2 the product*

$\mathcal{E}_T(\zeta - z)g(\zeta)$ *is well defined as an element of* $E[[z]]\left[\left[\zeta, \frac{1}{\zeta}\right]\right]$, *and* $w(z)$ *can be rewritten in an integral form*:

$$w(z) = \frac{1}{2\pi i}\oint \mathcal{E}_T(\zeta - z)g(\zeta)d\zeta .$$

Proof. First we need to show that $\mathcal{E}_T(\zeta - z)g(\zeta)$ exists as an element of $E[[z]]\left[\left[\zeta, \frac{1}{\zeta}\right]\right]$. We have

$$\mathcal{E}_T(\zeta - z) = \sum_{n=0}^{\infty} n!\frac{T^n}{\zeta^{n+1}}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \dots\right)^{n+1} = \sum_{n=0}^{\infty} n!T^n \sum_{j=n}^{\infty} C_j^m \frac{z^{j-n}}{\zeta^{j+1}} =$$

$$= \sum_{j=0}^{\infty}\left(\sum_{n=0}^{j} n!C_j^m z^{-n}T^n\right)\frac{z^j}{\zeta^{j+1}} = \sum_{j=0}^{\infty}\left(\sum_{n=0}^{j}\frac{j!}{(j-n)!z^n}T^n\right)\frac{z^j}{\zeta^{j+1}}.$$

If $g(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$, then the product $\mathcal{E}_T(\zeta - z)g(\zeta)$ can be formally rewritten as follows:

$$\mathcal{E}_T(\zeta - z)g(\zeta) = \left(\sum_{j=0}^{\infty}\left(\sum_{n=0}^{j}\frac{j!}{(j-n)!z^n}T^n\right)\frac{z^j}{\zeta^{j+1}}\right)\cdot\left(\sum_{k=0}^{\infty} c_k\,\zeta^k\right) =$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} c_k \sum_{m=0}^{n+k}\frac{(n+k)!z^{n+k-m}}{(n+k-m)!}T^m\right)\frac{1}{\zeta^{n+1}} +$$

$$+ \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} c_{k+n+1}\sum_{m=0}^{k}\frac{k!z^{k-m}}{(k-m)!}T^m\right)\zeta^n.$$

One can use estimations for the operator T and $g(z)$ to show that the coefficients at each term ζ^n, $n \in \mathbb{Z}$ are convergent series. We check this only for the coefficient at the term $\frac{1}{\zeta}$, which we are mostly interested in. Write out separately this coefficient:

$$\sum_{k=0}^{\infty} c_k\left(\sum_{m=0}^{k}\frac{k!T^m}{(k-m)!}\right)z^{k-m} = \sum_{m=0}^{\infty} T^m \sum_{k=m}^{\infty} c_k \frac{k!}{(k-m)!}z^{k-m} =$$

$$= \sum_{j=0}^{\infty} T^m g^{(m)}(z) = w(z),$$

and

$$\sum_{k=0}^{\infty} c_k \sum_{m=0}^{k} \frac{k!}{(k-m)!} T^m z^{k-m} = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} c_{k+m} \frac{(k+m)!}{k!} T^m \right) z^k .$$

To validate the transformations above it is sufficient to show

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left\| T^m \right\| \left| c_{k+m} \right| \frac{(k+m)!}{k!} |z|^{k-m} < +\infty ,$$

for all $z \in \mathbb{C}$. Let σ_1 and r be such numbers that $\sigma < \sigma_1 < r < \sigma_0$. As $\rho(T) = \frac{1}{\sigma_0}$ and g is an entire function of the exponential type σ, then $\left\| T^m \right\| \le \frac{M_1}{r^m}$ and $c_{k+m} \le \frac{M_2 \sigma_1^{k+m}}{(k+m)!}$ for certain $M_1, M_2 > 0$ and all integers m, k. Therefore,

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left\| T^m \right\| \left| c_{k+m} \right| (k+m)!}{k!} |z|^k \le \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{M_1 M_2}{r^m} \cdot \frac{\sigma_1^{k+m}}{k!} |z|^k \le$$

$$\le M_1 M_2 \sum_{m=0}^{\infty} \frac{\sigma_1^m}{r^m} \sum_{k=0}^{\infty} \frac{\sigma_1^k}{k!} |z|^k = \frac{M_1 M_2}{1 - \frac{\sigma_1}{r}} e^{\sigma_1 |z|}, \quad z \in \mathbb{C} \tag{11}$$

where $\sigma_1 < r$, so $\sum_{m=0}^{\infty} \frac{\sigma_1^m}{r^m}$ converges. Thus, $\sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} c_{k+m} \frac{(k+m)!}{k!} T^m \right) z^k$ is an entire function, and we obtain

$$\frac{1}{2\pi i} \oint \mathcal{E}_T (\zeta - z) g(\zeta) d\zeta = w(z) .$$

Finally, point out that Evaluation (11) reveals again that $w(z)$ is an entire function of exponential type at most σ. The theorem is proved.

Let us give some examples.

To our opinion, Theorem 2 is even interesting in one-dimensional case.

Example 8. Let $E = \mathbb{C}$ and $A = I$. Consider the differential equation $w' = w + f(z)$. If $f(z)$ is an entire function of exponential type $\sigma < 1$, then this equation has a unique entire solution of exponential type σ: $w(z) = -\sum_{n=0}^{\infty} f^{(n)}(z)$ and this solution continuously depends on f in the topology of the space $\tilde{E}_1$.

Example 9. Consider the following equation of forced oscillations $\ddot{x} + \omega^2 x = f(t)$, where $\omega > 0$ and $f(t)$ is the trace on the real axes of an entire function of exponential type σ. Passing to the system of equations of the first order in Theorem 2 we obtain that under $\sigma < \omega$ this equation has a unique solution, which can be extended to an entire function of exponential type σ, $x(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\omega^{2k+2}} f^{(2k)}(t)$.

Example 10. Let E be a Hilbert space, A be a closed normal operator on E with discrete spectrum and $0 \notin \sigma(A)$. Let $\{e_k\}$ be the orthonormalized eigenbasis for A, $Ae_k = \lambda_k e_k$ and $\lambda_k \to \infty$. If $|\lambda_1| = \min_k |\lambda_k|$ and $f : \mathbb{C} \to E$, $f(z) = \sum_k f_k(z) e_k$ is an entire function and the exponential type of f is less than $|\lambda_1|$, then Equation (1) $w' = Aw + f(z)$ has the following unique entire solution, for which the exponential type is less than $|\lambda_1|$: $w(z) = -\sum_{n=0}^{\infty} \left(\sum_k \lambda_k^{-(n+1)} f_k^{(n)}(z) e_k \right)$.

Example 11. Let $E = C[0,1]$, $D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$. Then the operator A is invertible, $(A^{-1}h)(x) = \int_0^1 G(x,y) h(y) dy$, where G is the Green function of corresponding boundary problem, and $\rho(A^{-1}) = 1/\pi^2$. In this case

$$\left(A^{-(n+1)} h\right)(x) = \int_0^1 G_{n+1}(x,y) h(y) dy,$$

where $G_1(x,y) = G(x,y)$, $G_{n+1}(x,y) = \int_0^1 G_n(x,s) G(s,y) ds$.

In this example by transition to real axes Equation (1) has the form of the heat equation on $(0,1)$ with zero boundary conditions

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(t,x), \quad t \in \mathbb{R}, x \in (0,1) \\ w(t,0) = w(t,1) = 0 \end{cases} \tag{12}$$

If $f(t,x) = \sum_{n=0}^{\infty} c_n(x) t^n$, where $c_n \in C[0,1]$ and $\overline{\lim_{n\to\infty}} \sqrt[n]{n! \|c_n\|} < \frac{1}{\pi^2}$, then the problem (12) has the solution $w(t,x) = -\sum_{n=0}^{\infty} \int_0^1 G_{n+1}(x,y) \frac{\partial^n f}{\partial t^n}(t,y) dy$.

Example 12. Let $E = C[0,1]$, $A = \frac{d}{dx}$ and $D(A) = \{u \in C^1[0,1] : u(0) = 0\}$.

Then $\left(A^{-1}h\right)(x)=\int_0^x h(y)dy$, $\left(A^{-(n+1)}h\right)(x)=\frac{1}{n!}\int_0^x (x-y)^n h(y)dy$ and $\rho\left(A^{-1}\right)=0$.

By transition to real axes Equation (1) has the form

$$\begin{cases} \frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}+f(t,x), \quad t\in\mathbb{R}, x\in(0,1) \\ w(t,0)=0 \end{cases} \tag{13}$$

If in the second variable f can be extended to an entire function of exponential type, then in this class of functions Problem (13) has the unique solution

$$w(t,x)=-\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^x (x-y)^n \frac{\partial^n f}{\partial t^n}(t,y)dy=-\int_0^x f(t+x-y,y)dy.$$

It is important to note, that Problem (13) has only zero solution for the homogeneous equation even in class of continuously differentiable functions. In particular, A is not a Hille–Yosida operator (see [5], Section 3.5).

Example 13. Let us consider the same space $E=C[0,1]$ and let $A=\frac{d^2}{dx^2}$ with $D(A)=\{u\in C^2[0,1]: u(0)=u'(0)=0\}$. Then A is invertible, $\left(A^{-1}h\right)(x)=\int_0^x (x-y)h(y)dy$ is the square of the integration operator and $\left(A^{-(n+1)}h\right)(x)=\frac{1}{(2n+1)!}\int_0^x (x-y)^{2n+1} h(y)dy$.

Equation (1) has the form of the heat equation with special boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}+f(t,x), \quad t\in\mathbb{R}, x\in(0,1) \\ u(t,0)=0 \\ \frac{\partial u(t,0)}{\partial x}=0. \end{cases} \tag{14}$$

If f of the variable x is extended to an arbitrary entire function of exponential type, then in this class of functions Problem (14) has a unique solution

$$u(t,x)=-\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_0^x (x-y)^{2n+1}\frac{\partial^n f}{\partial t^n}(t,y)dy, \quad t\in\mathbb{R}, x\in(0,1).$$

It is interesting to note that this formula is valid for any function which is holomorphic to t in a neighborhood of zero. For example, if $f(t,x)=\frac{1}{1-t}$, then we have the problem

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}+\frac{1}{1-t}, \quad t<1 \\ u(t,0)=0 \\ \frac{\partial u}{\partial x}(t,0)=0. \end{cases}$$

It can be shown that the problem has a unique holomorphic to t solution $u(t,x)=-\sum_{n=0}^{\infty}\frac{n!}{(2n+2)!}\frac{x^{2n+2}}{(1-t)^{n+1}}, t\in(-\infty,1), x\in[-1,1]$. It occurs because the operator A^{-1} is quasinilpotent and its Fredholm resolvent is an entire function of zero exponential type (see [18], Theorem 2.4).

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