BOUNDARY BEHAVIOUR OF SOLUTIONS TO THE BELTRAMI EQUATIONS

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1. Introduction. In this article we present applications of our results on the so-called lower Q–homeomorphisms in the monograph [1] to the study of the boundary behavior of solutions for the Beltrami equations with degeneration. Note that the existence theorems of homeomorphic solutions in the Sobolev class $W^{1,1}_{loc}$ have been established to many degenerate Beltrami equations (see, e.g., the papers [2–29], the recent monographs [1, 30] and the surveys [31, 32]).

Let D be a domain in the complex plane C, i.e. a connected and open subset of C, and let $\mu : D \to C$ be a measurable function with $|\mu(z)| < 1$ almost everywhere (a.e.) in D. The Beltrami equation is the equation of the form

$$f_{\bar{z}} = \mu(z) f_z , \tag{1.1}$$

where $f_{\bar{z}} = \bar{\partial} f = \frac{1}{2}(f_x + i f_y)$, $f_z = \partial f = \frac{1}{2}(f_x - i f_y)$, $z = x + iy$ and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the complex coefficient and

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{1.2}$$

the dilatation quotient for equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

A continuous mapping γ of an open subset Δ of the real axis R or a circle into D is called a dashed line (see, e.g., Section 6.3 in [1]). Recall that every open set Δ in R consists of a countable collection of mutually disjoint intervals. This is the motivation for the term.

Given a family Γ of dashed lines γ in complex plane C, a Borel function $\rho : C \to [0, \infty]$ is called admissible for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int_\gamma \rho \, ds \geq 1 \tag{1.3}$$

for every $\gamma \in \Gamma$. The (conformal) modulus of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_C \rho^2(z) \, dm(z), \tag{1.4}$$

where $dm(z)$ corresponds to the Lebesgue measure in C. We say that a property P holds for a.e. $\gamma \in \Gamma$ if a subfamily of all lines in Γ for which P fails has the modulus zero (cf. [33]). Later on, we also say that a Lebesgue measurable function $\rho : C \to [0,\infty]$ is extensively admissible for Γ, write $\rho \in \operatorname{ext\,adm} \Gamma$, if (1.3) holds for a.e. $\gamma \in \Gamma$ (see, e.g. Section 9.2 in [1]).

The following concept in [34], see also [1], was motivated by Gehring's ring definition of quasiconformality [35]. Given domains D and D' in $\overline{C} = C \cup \{\infty\}, z_0 \in \overline{D} \setminus \{\infty\}$ and a measurable function $Q : D \to (0,\infty)$, we say that a homeomorphism $f : D \to D'$ is a lower Q–homeomorphism at the point z_0 if

$$M(f\Sigma_\varepsilon) \geq \inf_{\rho \in \operatorname{ext\,adm} \Gamma} \int_{D \cap R_\varepsilon} \frac{\rho^2(z)}{Q(z)} dm(z) \tag{1.5}$$

for every ring

$$R_\varepsilon = \{z \in \overline{C} : \varepsilon < |z - z_0| < \varepsilon_0\}, \qquad \varepsilon \in (0,\varepsilon_0),\ \varepsilon_0 \in (0,d_0),$$

where

$$d_0 = \sup_{z \in D} |z - z_0|,$$

and Σ_ε denotes the family of all intersections of the circles

$$S(r) = S(z_0, r) = \{z \in C : |z - z_0| = r\}, \qquad r \in (\varepsilon, \varepsilon_0),$$

with the domain D.

This notion can be extended to the case $z_0 = \infty \in \overline{D}$ in the standard way by applying the inversion T with respect to the unit circle in $\overline{C}$, $T(z) = z/|z|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a lower Q–homeomorphism at $\infty \in \overline{D}$ if $F = f \circ T$ is a lower Q_*–homeomorphism with $Q_* = Q \circ T$ at 0. We also say that a homeomorphism $f : D \to \overline{C}$ is a lower Q–homeomorphism in ∂D if f is a lower Q–homeomorphism at every point $z_0 \in \partial D$.

We show that each homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (1.1) is a lower Q–homeomorphism with $Q(z) = K_\mu(z)$ and, thus, the whole theory of the boundary behaviour in [34] (see also Chapter 9 in [1]), can be applied to such solutions. In other words, in the plane this holds for all homeomorphisms with finite distortion by Iwaniec (see, e.g., related references in the monographs [1, 31]). This fact is important for the study of the boundary value problems, in particular, of the Dirichlet problem to the Beltrami equations with degeneration (see, e.g., [36]).

2. On weakly flat and strongly accessible boundaries. Recall first of all the following topological notion. A domain $D \subset C$ is said to be *locally connected at a point* $z_0 \in \partial D$ if, for every neighbourhood U of the point z_0, there is a neighbourhood $V \subseteq U$ of z_0 such that $V \cap D$ is connected. Note that every Jordan domain D in C is locally connected at each point of ∂D (see, e.g., [37, p. 66]).

We say that ∂D is *weakly flat at a point* $z_0 \in \partial D$ if, for every neighbourhood U of the point z_0 and every number $P > 0$, there is a neighbourhood $V \subset U$ of z_0 such that

$$M(\Delta(E, F; D)) \geq P \tag{2.1}$$

for all continua E and F in D intersecting ∂U and ∂V. Here and later on, $\Delta(E, F; D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{C}$ connecting E and F in D, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$.We say that the boundary ∂D is *weakly flat* if it is weakly flat at every point in ∂D.

We also say that a point $z_0 \in \partial D$ is *strongly accessible* if, for every neighbourhood U of the point z_0, there exist a compactum E in D, a neighbourhood $V \subset U$ of z_0 and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geq \delta \tag{2.2}$$

for all continua F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *strongly accessible* if every point $z_0 \in \partial D$ is strongly accessible.

Here, in the definitions of strongly accessible and weakly flat boundaries, one can take U and V only as balls (closed or open) centred at z_0 or only neighbourhoods of z_0 in another fundamental system of neighbourhoods of z_0. These conceptions can also be extended in a natural way to the case of $\overline{C}$ and $z_0 = \infty$. Then we must use the corresponding neighbourhoods of ∞.

It is easy to see that if a domain D in C is weakly flat at a point $z_0 \in \partial D$, then the point z_0 is strongly accessible from D. Moreover, it was proved by us that if a domain D in C is weakly flat at a point $z_0 \in \partial D$, then D is locally connected at z_0 (see, e.g., Lemma 5.1 in [36] or Lemma 3.15 in [1]).

The notions of strong accessibility and weak flatness at boundary points of a domain in C introduced in [38] are localizations and generalizations of the corresponding notions introduced in [39, 40] (cf. with the properties P_1 and P_2 by Väisälä [41] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki [42]). Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary. In particular, recently we have proved the following significant statements (see either Theorem 10.1 (Lemma 6.1) in [34] or Theorem 9.8 (Lemma 9.4) in [1]).

Proposition 2.1 Let D and D' be bounded domains in C, $Q: D \to (0,\infty)$ a measurable function and $f: D \to D'$ a lower Q–homeomorphism in ∂D. Suppose that the domain D is locally connected on ∂D and that the domain D' has a (strongly accessible) weakly flat boundary. If

$$\int_0^{\delta(z_0)} \frac{dr}{\|Q\|_1(z_0,r)} = \infty \qquad \forall z_0 \in \partial D \tag{2.3}$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z\in D} |z - z_0|$ and

$$\|Q\|_1(z_0,r) = \int_{D\cap S(z_0,r)} Q(z)\,ds\,,$$

then f has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ in $\overline{C}$ that maps $\overline{D}$ (into) onto $\overline{D}'$.

Here as usual $S(z_0,r)$ denotes the circle $|z - z_0| = r$.

A domain $D \subset C$ is called a *quasiextremal distance domain* (QED–domain) [43] if

$$M(\Delta(E,\, F; \overline{C})) \le K \cdot M(\Delta(E,\, F; D)) \tag{2.4}$$

for some $K \ge 1$ and all pairs of nonintersecting continua E and F in D.

It is well known (see, e.g., Theorem 10.12 in [41]) that

$$M(\Delta(E,\, F; C)) \ge \frac{2}{\pi} \log \frac{R}{r} \tag{2.5}$$

for any sets E and F in C intersecting all the circles $S(z_0,\rho)$, $\rho \in (r,R)$. Hence a QED–domain has a weakly flat boundary. One example in [1, Section 3.8] shows that the inverse conclusion is not true even among simply connected plane domains.

A domain $D \subset C$ is called a uniform domain if each pair of points z_1 and $z_2 \in D$ can be joined with a rectifiable curve γ in D such that

$$s(\gamma) \le a \cdot |z_1 - z_2| \tag{2.6}$$

and

$$\min_{i=1,2} s(\gamma(z_i, z)) \le b \cdot d(z, \partial D) \tag{2.7}$$

for all $z \in \gamma$ where $\gamma(z_i, z)$ is the portion of γ bounded by z_i and z [44]. It is known that every uniform domain is a QED–domain but there exist QED–domains that are not uniform [43]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED–domains as well as domains with weakly flat boundaries.

Also we denote by $C(X,f)$ the *cluster set* of the mapping $f:D\to\overline{C}$ for a set $X\subset\overline{D}$,

$$C(X,f):=\left\{w\in\overline{C}: w=\lim_{k\to\infty} f(z_k),\ \ z_k\to z_0\in X,\ z_k\in D\right\}. \tag{2.8}$$

Note that the inclusion $C(\partial D,f)\subseteq\partial D'$ holds for every homeomorphism $f:D\to D'$ (see, e.g., Proposition 13.5 in [1]).

3. The theory of the boundary behaviour. The following theorem is key for applications of the theory of the lower Q–homeomorphisms, see [45].

Theorem 3.1 *Let* f *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1). Then* f *is a lower* Q*–homeomorphism at each point* $z_0\in\overline{D}$ *with* $Q(z)=K_\mu(z)$.

In view of Theorem 3.1, we have by Lemma 6.1 in [34] or Lemma 9.4 in [1] the next statement.

Lemma 3.1 *Let* D *and* D' *be domains in* C, $z_0\in\partial D$, *and let* $f:D\to D'$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1). Suppose that the domain* D *is locally connected at* $z_0\in\partial D$ *and* $\partial D'$ *is strongly accessible at least at one point of the cluster set* $C(z_0,f)$. *If*

$$\int_0^{\varepsilon_0}\frac{dr}{\|K_\mu\|_1(z_0,r)}=\infty, \tag{3.1}$$

where $0<\varepsilon_0<d_0=\sup\limits_{z\in D}|z-z_0|$, *and*

$$\|K_\mu\|_1(z_0,r)=\int\limits_{D(z_0,r)}K_\mu ds \tag{3.2}$$

is the L_1*–norm of* K_μ *over*

$$D(z_0,r)=\{z\in D:|z-z_0|=r\}=D\cap S(z_0,r).$$

Then f *extends to* z_0 *by continuity in* $\overline{C}$.

The basis for the proof on extending the inverse mappings of homeomorphic $W^{1,1}_{loc}$ solutions to the Beltrami equation (1.1) is the following lemma on the cluster sets.

Lemma 3.2 *Let* D *and* D' *be domains in* C, z_1 *and* z_2 *be distinct points in* ∂D, $z_1\neq\infty$, *and let* $f:D\to D'$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1). Suppose that the function* K_μ *is integrable on the dashed lines*

$$D(r)=\{z\in D:|z-z_1|=r\}=D\cap S(z_1,r) \tag{3.3}$$

for some set E *of numbers* $r<|z_1-z_2|$ *of a positive linear measure. If* D *is locally connected at* z_1 *and* z_2 *and* $\partial D'$ *is weakly flat, then*

$$C(z_1,f)\cap C(z_2,f)=\varnothing \tag{3.4}$$

In view of Theorem 3.1, Lemma 3.2 follows from Lemma 9.1 in [34] or Lemma 9.5 in [1].
As an immediate consequence of Lemma 3.2, we have the following statement.

Theorem 3.2 *Let* D *and* D' *be domains in* $\mathbb{C}$, D *locally connected on* ∂D *and* $\partial D'$ *weakly flat. If* $f:D\to D'$ *is a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1) with* $K_\mu\in L^1(D)$, *then* f^{-1} *has an extension to* $\overline{D'}$ *by continuity in* $\overline{\mathbb{C}}$.

Remark 3.1. Simultaneously, there exist examples showing that any power of integrability of K_μ cannot guarantee the same for the direct mappings f, see, e.g., Proposition 6.3 in [1]. It is clear also that it is even sufficient to assume in Theorem 3.2 that K_μ is integrable only in a neighbourhood of ∂D.

Moreover, in view of Theorem 3.1, we obtain by Theorem 9.2 in [34] or Theorem 9.7 in [1] the following conclusion.

Theorem 3.3 *Let* D *and* D' *be domains in* $\mathbb{C}$, D *locally connected on* ∂D *and* $\partial D'$ *weakly flat, and let* $f:D\to D'$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1) with the coefficient* μ *such that the condition*

$$\int_0^{\delta(z_0)}\frac{dr}{\|K_\mu\|_1(z_0,r)}=\infty,\qquad \forall\, z_0\in\partial D \tag{3.5}$$

holds with some $\delta(z_0)\in(0,d(z_0))$ *where* $d(z_0)=\sup\limits_{z\in D}|z-z_0|$ *and* $\|K_\mu\|_1(z_0,r)$ *is defined in (3.2). Then there is an extension of* f^{-1} *by continuity to* $\overline{D'}$ *in* $\overline{\mathbb{C}}$.

Combining Lemma 3.1 and Theorem 3.3, we obtain the following statements.

Theorem 3.4 *Let* D *and* D' *be bounded domains in* $\mathbb{C}$ *and let* $f:D\to D'$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1). Suppose that the domain* D *is locally connected on* ∂D *and that the domain* D' *has a weakly flat boundary. If the condition (3.5) holds, then* f *has a homeomorphic extension to* $\overline{D}$.

In particular, as a consequence of Theorem 3.4, we obtain the following generalization of the well-known Gehring–Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED domains, see [43].

Corollary 3.1 *Let* D *and* D' *be bounded domains with weakly flat boundaries in* $\mathbb{C}$ *and let* $f : D \to D'$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1). If the condition (3.5) holds, then* f *has a homeomorphic extension to* $\overline{D}$.

Integral conditions of the type

$$\int_D \Phi(K(x))\,dm(x) < \infty \tag{3.6}$$

are often applied in the mapping theory (see, e.g. [4, 18, 46–53].
Combining Theorem 3.1 in the paper [54] and Lemma 3.1 and Theorem 3.4 above, we come to the following statement.

Theorem 3.5. *Let* D *and* D' *be bounded domains in* $\mathbb{C}$ *such that* D *is locally connected at* ∂D *and* D' *has a weakly flat (strongly accessible) boundary. Suppose that* $f : D \to D'$ *is a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation (1.1) such that*

$$\int_D \Phi(K_\mu(z))\,dm(z) < \infty \tag{3.7}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$. *If*

$$\int_\delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{3.8}$$

for some $\delta > \Phi(0)$, *then* f *has a homeomorphic (continuous) extension* $\overline{f}$ *to* $\overline{D}$ *that maps* $\overline{D}$ *onto (into)* $\overline{D'}$.

Remark 3.2 In particular, the conclusion on homeomorphic extension is valid for many regular domains D and D' as smooth, Lipshitz, convex, quasiconvex, uniform, quasiextremal distance by Gohring–Martio. The example in [55] shows that the condition (3.8) is not only sufficient but also necessary for continuous extension of f to the boundary.

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