ABOUT CONSTRUCTION OF EXACT SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH GIVEN CLASS OF DIFFERENTIABILITY IN DOMAINS OF COMPLEX SHAPES

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In [1] the general method of construction of exact solutions of boundary value problems for differential equations with 2-nd order partial derivatives of elliptic type in the domains of complex shape is offered. The method substantially uses operators of spline–interlineation functions of two variables. In this work the method of spline–interlineation of functions of two variables is used for construction and research of exact solutions of not uniform boundary value problems for differential equations of elliptic type, which have specified class of differentiability in domains of complex shape. Actuality of work is explained by arising necessity of testing of the offered algorithms during development of new numerical methods of solution of boundary value problems in domains of complex shape, where testing could be comfortably conducted on tasks with the known solutions, which allow to conduct the analysis of approaching accuracy. This work considers examples for domain, which has form of channel section, and for a Z–similar domain.

Analysis of the known results. It is known that the general method of construction of functions, which fulfill not uniform boundary conditions of Dirichlet, Neumann, and mixed, can be realized using RFM– method (R–functions) offered by V. L. Rvachov [2–4]. But, as work [5] reports, use of the structural method built using R–functions of V. L. Rvachov leads to arising some problems: problem of angular points*;* problem of continuation of tracks of functions and their normal derivatives from boundary to internal points of domain G of integration with saving of class of differentiability $C^r(G)$*;* problem of change of boundary conditions type in some arbitrary points of boundary; problem of construction of structures of approximate solutions with given tracks on M lines, if $m\,(m \geq 3)$ from them are intersected in one point, etc. Above-mentioned problems can be successfully decided by the methods of interlineation of function of two variables, interflетation of functions of three or more variables [5–6]. Therefore, R–functions method not always can be applied for the method of construction of exact solutions proposed in this work.

Problem formulation. On the basis of spline–interlinations of functions of two variables we will build the exact solution of boundary value problem for self-conjugating elliptic operator of 2-nd order:

$$L_{2m}u(x,y) = f(x,y), \quad (x,y) \in \Omega \tag{1}$$

$$\left.\frac{\partial^p u}{\partial \nu^p}\right|_{\partial\Omega} = g_p(x,y), \quad p = \overline{0, m-1} \tag{2}$$

where ν is normal directed inwards the domain Ω,

$$L_{2m}=\sum_{r+s\le m}(-1)^{r+s}D^{(r,s)}(a_{r,s}(x,y)D^{(r,s)}).$$

Theorem. *Let right part of differential equation (1) be determined by equality* $f(x,y)=L_{2m}U(x,y)$, *where* $U(x,y)$ *is determined by corresponding formula of interlineation in each of elements of partitioning of domain of integration* Ω *by lines* $x=x_k,\ y=y_\ell\ (k=\overline{0,n_1+1},\ \ell=\overline{0,n_2+1})$ *(rectangular, rectangular with one curvilinear, generally speaking, side, which belongs to boundary of domain* Ω *of integration, trigonal, trigonal with one curvilinear, generally speaking, hypotenuse, which belongs to the boundary of domain of integration). Then there are such functions* $\phi_{\ell,q}(x),\ \psi_{k,p}(y),\ 0\le p,q\le r$ *and constants* $u^{(p,q)}(x_k,y_\ell)$, *at which* $U(x,y)\in C^r\left(\overline{D}\right), r\ge 1$ *is the exact solution of boundary value problem with given right part and boundary conditions of Dirichlet, or it is the exact solution of boundary value problem with the given right part and boundary conditions of Dirichlet or Neumann or mixed boundary conditions. Depending on the type of boundary conditions and the* r *value functions* $\phi_{\ell,q}(x),\ \psi_{k,p}(y),\ 0\le p,q\le r$ *are chosen as even to specified boundary functions.*

Description of method. Let assume that domain $\Omega\subset[a,b]\times[c,d]$ is the polygon limited by lines, which are parallel by coordinate axes of the Oxy system. We will divide her into rectangular elements $\Pi_{k,\ell}=[x_{k-1},x_k]\times[y_{\ell-1},y_\ell]$ by lines

$x=x_k,\ k=\overline{1,n_1},\ a=x_0<x_1<x_2<...<x_{n_1}=b;$

$y=y_\ell, \ell=\overline{1,n_2},\ c=y_0<y_1<y_2<...<y_{n_2}=d.$

Let assign

$$H_{i,j}(x)=\begin{cases}\sum_{r=0}^{m-j-1}\frac{1}{r!j!}\left[\frac{(x-x_i)^m}{\omega(x)}\right]^{(r)}_{x=x_i}\frac{\omega(x)}{(x-x_i)^{m-j-r}}, & x_{k-1}\le x\le x_k\\ 0, & x<x_{k-1}, x>x_k\end{cases}, \quad (3)$$

$$\omega(x)=(x-x_{k-1})^m(x-x_k)^m,\quad i=k-1,k;\ j=\overline{0,m-1};$$

similarly

$$h_{p,q}(y)=\begin{cases}\sum_{s=0}^{m-q-1}\frac{1}{s!q!}\left[\frac{(y-y_p)^m}{\omega(y)}\right]^{(s)}_{y=y_p}\frac{\omega(y)}{(y-y_p)^{m-q-s}}, & y_{\ell-1}\le y\le y_\ell\\ 0, & y<y_{\ell-1}, y>y_\ell\end{cases} \quad (4)$$

$$\omega(y)=(y-y_{\ell-1})^m(y-y_\ell)^m,\quad p=\ell-1,\ell;\ q=\overline{0,m-1};$$

$$\phi_{q,j}(x)=\left.\frac{\partial^j u}{\partial y^j}\right|_{y=y_q},\ \psi_{p,i}(y)=\left.\frac{\partial^i u}{\partial x^i}\right|_{x=x_p},q=\overline{0,n_2},p=\overline{0,n_1},\ i,j=\overline{0,m-1};$$

$$u_{pqij}=\left.\frac{\partial^{i+j}u}{\partial x^i\partial y^j}\right|_{(x_p,y_q)}$$

and solution $u(x,y)$ of problem (1)–(2) in rectangle $\Pi_{k,\ell}$ we will search in the form of

$$u(x,y)=u_{k,l}(x,y)=\sum_{p=k-1}^{k}\sum_{r=0}^{m-1}H_{p,r}(x)\psi_{p,r}(y)+\sum_{q=\ell-1}^{\ell}\sum_{s=0}^{m-1}h_{q,s}(y)\phi_{q,s}(x)-$$
$$-\sum_{p=k-1}^{k}\sum_{q=\ell-1}^{\ell}\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}u_{pqrs}H_{p,r}(x)h_{q,s}(y)\ (x,y)\in\Pi_{k,\ell}\,. \qquad (5)$$

Let mark that functions $H_{p,r}(x),\ h_{q,s}(y)$, which are determined by equality (3)–(4), satisfy to next conditions.

$$H^{(s)}_{p,r}(x_t)=\delta_{p,t}\delta_{s,r},\ h^{(s)}_{p,r}(y_t)=\delta_{p,t}\delta_{s,r}\quad;\ 0\le r,s\le m-1 \qquad (6)$$

(δ_{ij} – Kronecker's delta **).**

As $\phi_{q,s}(x)$ and $\psi_{p,r}(y)$ are tracks of the sought solution and his unmixed derivatives to $m-1$ order on key lines $x=x_p\ (p=\overline{0,n_1})$**,** $y=y_q\ (q=\overline{0,n_2})$**,** and constants $u_{pqij}\ (p=\overline{0,n_1},\ q=\overline{0,n_2},\ i,j=\overline{0,m-1})$ are the values of the sought solution after and his partial derivatives in key points $(x_p,y_q)\ (p=\overline{0,n_1},\ q=\overline{0,n_2})$**,** the S. M. Nikolsky conditions should be met (conditions of conjunction):

$$\phi^{(r)}_{q,s}(x_p)=\psi^{(s)}_{p,r}(y_q)=u_{pqrs}=u^{(r,s)}(x_p,y_q),\ 0\le r,s\le m-1.$$

By direct verification we set that, with an account (3)–(6), function $u_{k,l}(x,y)\ \ (k=\overline{1,n_1},\ l=\overline{1,n_2})$ satisfies conditions

$$\left.u^{(p)}_{k,l}(x,y)\right|_{x=x_i}=\psi_{i,p}(y),\ i=\overrightarrow{k-1,k},\ p=\overline{0,m-1};$$

$$\left.u^{(q)}_{k,\ell}(x,y)\right|_{y=y_j}=\phi_{j,q}(x),\ j=\overrightarrow{\ell-1,\ell},\ q=\overline{0,m-1}\,\mathbf{.}$$

Then function $u(x,y)=u_{k,l}(x,y),\ (x,y)\in\Pi_{k,l},\ \bigcup_{k,l}\Pi_{k,l}=\Omega$ satisfies conditions (2), and it is the solution of equation (1), if

$$f(x,y)=L_{2m}u_{k,l}(x,y),\ (x,y)\in\Pi_{k,l},\bigcup_{k,l}\Pi_{k,l}=\Omega$$

Let mark that this solution belongs to space $C^{m-1,m-1}(\Omega)$.

It is necessary to notice that in the case of construction of exact solution with the formula (5) there is the question of choice of functions $\phi_{q,s}(x)$ and

$\psi_{p,r}(y)$ and constants u_{pqrs}, which should be connected to each other by the conditions of compatibility in the points of intersections of lines of rectangulations regardless of their values in other points, and from the choice of constants u_{pqrs}. Thus, by replacing functions $\phi_{q,s}(x)$ and $\psi_{p,r}(y)$ by corresponding spline–interpolational operators and choosing constants u_{pqrs} definitely, we obtain one of possible algorithms of construction of exact solution, which will belong to the necessary class of differentiability.

Numerical realization.

1. This method could be generalized to the case of not uniform boundary conditions (boundary function in every point of boundary had continuous derivatives, and conditions of S. M. Nikolsky are met).

2. To build the solution, which belongs to the class $C^{1,1}(\Omega)$, spline–interlineation should be used with the application of cubic splines of the degre 3 of class C^1 or C^2.

3. Unknown tracks on the lines of rectangulation are chosen arbitrary with the account of only terms of conjugation (conditions of S. M. Nikolsky) on crossing of lines of rectangulation. Therefore, parameters in the knots (points of intersections of rectangulation lines) are considered arbitrary, and it is possible to simplify the kind of solution by choice of these points.

4. After construction of exact solution pursuant to the offered algorithm, it is possible to add to it expression $C_{i,j}\left[(x-x_i)(x-x_{i+1})(y-y_j)(y-y_{j+1})\right]^2$, where $C_{i,j}$ value will allow to change properties of exact solution in whole domain without changing the class of differentiability, in every rectangular element of partitioning.

For illustration of the offered method we will describe following examples of construction of exact solutions and their software implementation**.**

Example 1. Let find the exact solution of equation

$$\Delta u = f(x,y), \quad (x,y)\in\Omega, \tag{7}$$

that satisfies condition

$$u(x,y)\big|_{\partial\Omega} = 0 \;; \tag{8}$$

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

$$\Omega_1 = \{(x,y)\,|\,x_0 \le x \le x_1, y_0 \le y \le y_2\},$$

$$\Omega_2 = \{(x,y)\,|\,x_1 \le x \le x_2, y_0 \le y \le y_1\},$$

$$\Omega_3 = \{(x,y)\,|\,x_2 \le x \le x_3, y_0 \le y \le y_2\}.$$

According to the offered method, let break up domain Ω (“channel section”) to five rectangles**:**

$$\Pi_{1,1} = \{(x,y)\,|\,x_0 \le x \le x_1, y_0 \le y \le y_1\},$$

$$\Pi_{2,1} = \{(x,y) | x_1 \le x \le x_2, y_0 \le y \le y_1\},$$
$$\Pi_{3,1} = \{(x,y) | x_2 \le x \le x_3, y_0 \le y \le y_1\},$$
$$\Pi_{1,2} = \{(x,y) | x_0 \le x \le x_1, y_1 \le y \le y_2\},$$
$$\Pi_{3,2} = \{(x,y) | x_2 \le x \le x_3, y_1 \le y \le y_2\}$$

and let write down in each of them the solution of problem (7)–(8):

$$u(x,y) = u_{1,1}(x,y) = \sum_{k=0}^{1}\sum_{p=0}^{1} \phi_{k,p}(x) h_{k,p}(y) + \sum_{\ell=0}^{1}\sum_{q=0}^{1} \psi_{\ell,q}(y) H_{\ell,q}(x) -$$
$$-\sum_{k=0}^{1}\sum_{p=0}^{1}\sum_{\ell=0}^{1}\sum_{q=0}^{1} u^{(q,p)}(x_\ell, y_k) H_{\ell,q}(x) h_{k,p}(y),$$

$$u(x,y) = u_{2,1}(x,y) = \sum_{k=0}^{1}\sum_{p=0}^{1} \phi_{k,p}(x) h_{k,p}(y) + \sum_{\ell=1}^{2}\sum_{q=0}^{1} \psi_{\ell,q}(y) H_{\ell,q}(x) -$$
$$-\sum_{k=0}^{1}\sum_{p=0}^{1}\sum_{\ell=1}^{2}\sum_{q=0}^{1} u^{(q,p)}(x_\ell, y_k) H_{\ell,q}(x) h_{k,p}(y),$$

$$u(x,y) = u_{3,1}(x,y) = \sum_{k=0}^{1}\sum_{p=0}^{1} \phi_{k,p}(x) h_{k,p}(y) + \sum_{\ell=2}^{3}\sum_{q=0}^{1} \psi_{\ell,q}(y) H_{\ell,q}(x) -$$
$$-\sum_{k=0}^{1}\sum_{p=0}^{1}\sum_{\ell=2}^{3}\sum_{q=0}^{1} u^{(q,p)}(x_\ell, y_k) H_{\ell,q}(x) h_{k,p}(y),$$

$$u(x,y) = u_{1,2}(x,y) = \sum_{k=1}^{2}\sum_{p=0}^{1} \phi_{k,p}(x) h_{k,p}(y) + \sum_{\ell=0}^{1}\sum_{q=0}^{1} \psi_{\ell,q}(y) H_{\ell,q}(x) -$$
$$-\sum_{k=1}^{2}\sum_{p=0}^{1}\sum_{\ell=0}^{1}\sum_{q=0}^{1} u^{(q,p)}(x_\ell, y_k) H_{\ell,q}(x) h_{k,p}(y),$$

$$u(x,y) = u_{3,2}(x,y) = \sum_{k=1}^{2}\sum_{p=0}^{1} \phi_{k,p}(x) h_{k,p}(y) + \sum_{\ell=2}^{3}\sum_{q=0}^{1} \psi_{\ell,q}(y) H_{\ell,q}(x) -$$
$$-\sum_{k=1}^{2}\sum_{p=0}^{1}\sum_{\ell=2}^{3}\sum_{q=0}^{1} u^{(q,p)}(x_\ell, y_k) H_{\ell,q}(x) h_{k,p}(y).$$

Function

$$u(x,y) = \begin{cases} u_{1,1}(x,y), & (x,y) \in \Pi_{1,1} \\ u_{2,1}(x,y), & (x,y) \in \Pi_{2,1} \\ u_{3,1}(x,y), & (x,y) \in \Pi_{3,1} \\ u_{1,2}(x,y), & (x,y) \in \Pi_{1,2} \\ u_{3,2}(x,y), & (x,y) \in \Pi_{3,2} \end{cases} \tag{9}$$

belongs to the class $C^1(\Omega)$, and it is the exact solution of problem (7)–(8) at right part $f(x,y) = \Delta u(x,y) \in L_2(\Omega)$. On the fig. 1, the graph of the exact solution of problem is presented using the offered method.

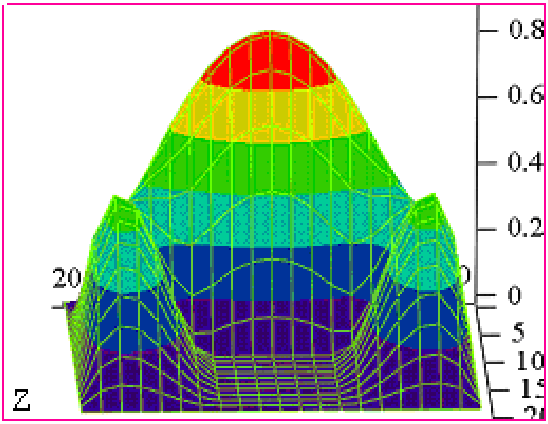

Fig. 1. Graph of exact solution (Example 1)

By direct calculations we check up meeting boundary condition and conditions of conjugation on lines $x = x_1$, $x = x_2$, $y = y_1$, which provide continuity of function $u(x,y)$ and its first order partial derivatives, provided that it is possible to take continuous and differentiable functions $\phi_{k,p}(x)$ and $\psi_{\ell,q}(y)$, $k = \overline{0,2}, \ell = \overline{0,3},\ p,q = \overline{0,1}$ as cube Hermitian splines.

Example 2. Let find the exact solution of problem (7)–(8) provided that domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$,

$$\Omega_1 = \{(x,y) | x_0 \le x \le x_1, y_0 \le y \le y_2\},$$

$$\Omega_2 = \{(x,y) | x_1 \le x \le x_2, y_1 \le y \le y_2\},$$

$$\Omega_3 = \{(x,y) | x_2 \le x \le x_3, y_1 \le y \le y_3\}$$

is a Z–profile.

Like above-mentioned, let break up the domain Ω to five rectangles:

$$\Pi_{1,1} = \{(x,y) | x_0 \le x \le x_1, y_0 \le y \le y_1\}$$

$$\Pi_{1,2} = \{(x,y) | x_0 \le x \le x_1, y_1 \le y \le y_2\},$$

$$\Pi_{2,2} = \{(x,y) | x_1 \le x \le x_2, y_1 \le y \le y_2\},$$

$$\Pi_{3,2} = \{(x,y) | x_2 \le x \le x_3, y_1 \le y \le y_2\},$$

$$\Pi_{3,3} = \{(x,y) | x_2 \le x \le x_3, y_2 \le y \le y_3\}$$

and let write down in each of them the solution of problem as

$$u(x,y)=u_{1,1}(x,y)=\sum_{k=0}^{1}\sum_{p=0}^{1}\phi_{k,p}(x)h_{k,p}(y)++\sum_{\ell=0}^{1}\sum_{q=0}^{1}\psi_{\ell,q}(y)H_{\ell,q}(x)-$$

$$-\sum_{k=0}^{1}\sum_{p=0}^{1}\sum_{\ell=0}^{1}\sum_{q=0}^{1}u^{(q,p)}(x_\ell,y_k)H_{\ell,q}(x)h_{k,p}(y),$$

$$u(x,y)=u_{1,2}(x,y)=\sum_{k=1}^{2}\sum_{p=0}^{1}\phi_{k,p}(x)h_{k,p}(y)+\sum_{\ell=0}^{1}\sum_{q=0}^{1}\psi_{\ell,q}(y)H_{\ell,q}(x)-$$

$$-\sum_{k=1}^{2}\sum_{p=0}^{1}\sum_{\ell=0}^{1}\sum_{q=0}^{1}u^{(q,p)}(x_\ell,y_k)H_{\ell,q}(x)h_{k,p}(y),$$

$$u(x,y)=u_{2,2}(x,y)=\sum_{k=1}^{2}\sum_{p=0}^{1}\phi_{k,p}(x)h_{k,p}(y)+\sum_{\ell=1}^{2}\sum_{q=0}^{1}\psi_{\ell,q}(y)H_{\ell,q}(x)-$$

$$-\sum_{k=1}^{2}\sum_{p=0}^{1}\sum_{\ell=1}^{2}\sum_{q=0}^{1}u^{(q,p)}(x_\ell,y_k)H_{\ell,q}(x)h_{k,p}(y),$$

$$u(x,y)=u_{3,2}(x,y)=\sum_{k=1}^{2}\sum_{p=0}^{1}\phi_{k,p}(x)h_{k,p}(y)+\sum_{\ell=2}^{3}\sum_{q=0}^{1}\psi_{\ell,q}(y)H_{\ell,q}(x)-$$

$$-\sum_{k=1}^{2}\sum_{p=0}^{1}\sum_{\ell=2}^{3}\sum_{q=0}^{1}u^{(q,p)}(x_\ell,y_k)H_{\ell,q}(x)h_{k,p}(y),$$

$$u(x,y)=u_{3,3}(x,y)=\sum_{k=2}^{3}\sum_{p=0}^{1}\phi_{k,p}(x)h_{k,p}(y)+\sum_{\ell=2}^{3}\sum_{q=0}^{1}\psi_{\ell,q}(y)H_{\ell,q}(x)-$$

$$-\sum_{k=2}^{3}\sum_{p=0}^{1}\sum_{\ell=2}^{3}\sum_{q=0}^{1}u^{(q,p)}(x_\ell,y_k)H_{\ell,q}(x)h_{k,p}(y).$$

Function

$$u(x,y)=\begin{cases}u_{1,1}(x,y), & (x,y)\in\Pi_{1,1}\\ u_{1,2}(x,y), & (x,y)\in\Pi_{1,2}\\ u_{2,2}(x,y), & (x,y)\in\Pi_{2,2}\\ u_{3,2}(x,y), & (x,y)\in\Pi_{3,2}\\ u_{3,3}(x,y), & (x,y)\in\Pi_{3,3}\end{cases} \tag{10}$$

belongs to the class $C^1(\Omega)$, and it is the exact solution of considered problem at right part $f(x,y)=\Delta u(x,y)\in L_2(\Omega)$, where $u(x,y)$ is determined by formula (10); its graph is presented on fig.2.

By direct calculations we check meeting boundary condition and conditions of conjugation on lines $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, which provide continuity of function and its first order partial derivatives, provided that it is possible to take continuous and differentiable functions $\phi_{k,p}(x),\psi_{\ell,q}(y),\ k,\ell=\overline{0,3},\ p,q=\overline{0,1}$.

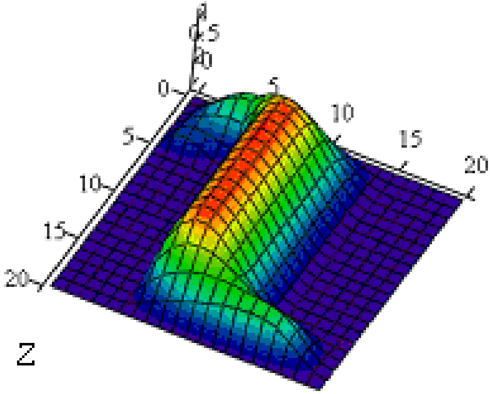

Fig. 2. Graph of exact solution (Example 2)

Conclusions. Therefore, offered in this work method of obtaining of exact solutions of boundary problems allows to obtain exact solutions, which belong to the class $C^1(\Omega)$ and have discontinuous function second derivatives (not mixed) on lines–sides of neighboring rectangular elements. As a result, right part of such boundary problem $f(x, y)$ will have the breaks of the first type between elements, i.e. this exact solution belongs to the class $W_2^1(\overline{\Omega})$. But the idea of the offered method allows to build the exact solutions, which belong to the class $C^r(\overline{\Omega})$ for a case, when $r \geq 2$. Next authors' work will be devoted to realization of these cases.

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