ABOUT CONSTRUCTION OF POLYNOMIALS $P_{2n+2m}(x)$ WITH PROPERTIES $P^{(s)}_{2n+2m}(\pm 1)=0$ $s=\overline{0,m-1}$ THE LEAST DEVIATING FROM A ZERO ON A SEGMENT $[-1, 1]$

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During the process of solving boundary value problems for differential equations with partial derivations in the areas of complicated form there appears the necessity to solve system of the linear algebraic equations of very large dimension in some practical situations. One of the ways to address this deficiency, i.e. decrease the order of solvable systems of equations, is the optimal choice of nodes used in the finite element method.

The choice of nodes of interpolation proves to be important for the general theory of approximation of functions of one variable by the interpolation polynomials of Lagrange. But in some cases an approximated function has zeros at predefined points (it refers, for example, to the case, when approximated function is the solution of multipoint boundary value problem for ordinary differential equations). In this case the authors failed to find the general theory for constructing such polynomials that have specified values (together with the derivatives to the order $(m-1)$) in the fixed system of points and deviate from zero the least. At the same time, taking into account a large influence of P.L. Chebishov's ideas concerning the construction of polynomials, which deviate from zero the least, over the development of the theory of approximation of functions by polynomials, the urgency of the task of constructing polynomials that deviate least from zero and satisfy the above mentioned limitations in a given system of points should be noted. This work offers such algorithm by use of the generalized Chebishov's alternance theorem. As an example a number of polynomials have been constructed and their application is demonstrated in approximation of functions and in the problem for optimum choice of nodes at solving boundary value problems.

Analysis of the latest research and publications. The authors of this paper have made a search of works, concerning the optimization of nodes in the method of finite elements [1–5], and concluded, that currently there exist no theorems, analogous to the P.L. Chebishov's theorems [6], concerning the best approximation of functions which are the solutions of boundary value problems.

As it is known [7], for the error of approximation of differentiable function by the interpolation polynomial of Lagrange

$$L_N y(x)=\sum_{k=0}^{N} y(X_k)\, l_{N,k}(x), \qquad (1)$$

where $l_{N,k}(x)=\prod_{\substack{j=0,\\ j\neq k}}^{N}\frac{x-X_j}{X_k-X_j}$ the following formula proves to be true

$$y(x)-L_N y(x)=\frac{y^{(N+1)}(\xi)}{(N+1)!}\prod_{j=0}^{N}(x-X_j),\quad a\le X_0<\xi<X_n\le b. \tag{2}$$

A very important conclusion can be made for the remainder term on the grounds of formula (2): the standard error in the metric is defined by the following inequality.

$$\left\|y(\cdot)-L_N y(\cdot)\right\|_{C[a,b]}\le\frac{M_{N+1}}{(N+1)!}\prod_{j=0}^{N}\left|x-X_j\right|,\quad a\le x\le b, \tag{3}$$

for all functions $y(x)$ with bounded $(N+1)$ derivative, $M_{N+1}=\max\limits_{a\le x\le b}\left|y^{(N+1)}(x)\right|$ i.e. the smallest error on the entire segment $[a,b]$ will depend on the choice of nodes $X_j\ \left(j=\overline{1,N}\right)$ of interpolation polynomial. Error is the smallest when the roots of Chebyshev's polynomials of the 1st kind with leading coefficient equal to one $T_n(x)=\cos(n\cdot\arccos x)$ are the interpolation nodes.

It is interesting from the practical point to solve the problem of choice of interpolation nodes in interpolation polynomial (1) for the case when in the form of (1) the approximation of boundary value problem solution is sought

$$Ay(x)=f(x),\quad a<x<b, \tag{4}$$

$$y^{(s)}(a)=0,\ y^{(s)}(b)=0,\qquad s=\overline{0,m-1}, \tag{5}$$

where $Ay(x)=\sum_{s=0}^{m}(-1)^s\frac{d^s}{dx^s}\left(p_s(x)\frac{d^s y}{dx^s}\right)$ and unknown parameters $C_1, C_2, \ldots, C_{N-1}$ in formula

$$L_N(x,X,C)=\sum_{k=1}^{N-1}C_k l_{N,k}(x) \tag{6}$$

are found on condition

$$J(X,C)=\int_a^b\left(\sum_{s=0}^{m}p_s(x)\left(\frac{d^s}{dx^s}L_N(x,X,C)\right)^2-2f(x)L_N(x,X,C)\right)dx\to\min_{X,C} \tag{7}$$

The authors are not familiar with the statement type (3) for the case when approximated function is the solution to boundary value problem for differential equations, that is set implicitly. However, if the solution of the problem (4) – (5) presented by using the Green function as

$$y(x)=\int_a^b G(x,t)f(t)dt,$$

it could be evaluated $y^{(N+1)}$ via the corresponding derivatives of the Green function. In this case we can use Lemma Cea [8] according to which the error of approximate solution for boundary value problem is bounded from above by the product of some constant, which depends on the differential operator of boundary value problem, by error of the best approximation of the exact solution of boundary value problem with used linear combinations of linearly independent functions (in our case – linear combinations of Lagrange's basic interpolation polynomials). But in the formula (3) it was not required that approximated function should be equal to zero at the ends of the interval for the balance.

E.Y. Zolotarev's works [9, 10] are concerned with other cases of restrictions on the polynomials that are the least different from zero. For example, the coefficient of x^{n-1} polynomial degree n fixed and other polynomial coefficients are found in such a way that its largest deviation from zero on the interval $[-1, 1]$ was the smallest.

As the optimal choice of nodes is the actuality problem this paper treats and solves the following task.

Problem formulation. To build and development the method of choice of polynomials nodes, which exactly satisfy the boundary conditions (5) at the ends of a given segment $[-1, 1]$ and the least deviate from zero on this segment.

Construction of polynomials $P_{2n+2m}(x)$ with the properties $P^{(s)}_{2n+2m}(\pm 1) = 0$, $s = \overline{0, m-1}$, that deviate least from zero on the segment $[-1, 1]$.

The method proposed in this paper is to construct polynomials that exactly satisfy the boundary conditions and at the same time the least deviation from zero. These polynomials are not PL Chebyshev's polynomials, since they must be equal to zero at the ends of the interval $[-1, 1]$. In order to find them we use the generalized Chebishov's alternance theorem.

Theorem 1 [11, p. 12]. *Let in an arbitrary normed linear space E any selected* $n+1$ *linearly independent elements* $g_0, g_1, ..., g_n$. *Then, for any* $x \in E$ *of "polynomials" of* $P_n(c;g)$ *type*

$$P_n(c;g) = \sum_0^n c_k g_k ,$$

where c_k *– are arbitrary real numbers, there is at least one "polynomial"* $P_n^*(c^*;g) = \sum_0^n c_k^* g_k$ *of best approximation element* x, *i.e. the one for which the correlation is performed*

$$\inf_{c_k} \left\| x - \sum_0^n c_k g_k \right\| = \left\| x - \sum_0^n c_k^* g_k \right\|.$$

Theorem 2 (on alternance) [10, p. 91]. *For the polynomial* $p_0(t)=\sum_{k=1}^{n+1} a_k t^{k-1}$, $a_{n+1}=1$ *to be the best approximation polynomial to the function* $x(\cdot)\in C[t_0,t_1]$ *in space* $C[t_0,t_1]$ *it is necessary and sufficient that the* $n+2$ *points* $t_0\le\tau_1<\ldots<\tau_{n+2}\le t_1$ *were found (points of alternance) in which the difference* $x(\cdot)-p_0(\cdot)$*, consistently alterning, takes its maximum and minimum value.*

For the case when the polynomial of best approximation must satisfy additional homogeneous boundary conditions, the following generalization of a theorem on alternance is true.

Theorem 3. *If the coefficients polynomial*

$$P_{2n+2m}(x)=\left(x^2-1\right)^m\left(x^{2n}+\sum_{k=0}^{n-1} a_{2k}x^{2k}\right) \tag{8}$$

to find in conditions $P_{2n+2m}(x_q)=(-1)^q\cdot\varepsilon$, $q=\overline{1,2n+1}$, *where* $\varepsilon=\max\limits_{-1<x<1}\left|P_{2n+2m}(x)\right|$, *then the resulting polynomial* $P_{2n+2m}(x)$ *is polynomial, which, together with derivatives up to* $m-1$ *order inclusive, equals to zero at* $x=\pm1$ *and least deviates from zero in the interval* $(-1,1)$*:*

$$\varepsilon=\varepsilon(a_0,a_2,\ldots,a_{2n-2})\to\min_{a_{2k},\,k=\overline{0,n-1}}.$$

The proof follows from the theorem on alternance [2].

Let us formulate a general algorithm for finding polynomial $P_{2n+2m}(x)$ of $2n+2m$ degree with a coefficient equal to one in the highest degree that satisfies the condition $P_{2n+2m}(-1)=P_{2n+2m}(1)=0$ and the least deviate from zero in the norm $C[-1,1]$ on the interval $[-1,1]$:

- looking for a polynomial $P_{2n+2m}(x)$ in (8);
- believe that the required polynomial has extremum (with the same absolute value) at the points located symmetrically relating to zero, and at a point $x=0$, i.e. it is a pair one by construction, the total number of extremum points equals $2n+1$;
- unknown coefficients $a_{2k}\,(k=\overline{0,n-1})$ are found from on condition

$$P_{2n+2m}(x_q)=(-1)^q\varepsilon,\ q=\overline{1,2n+1}\ (\varepsilon>0), \tag{9}$$

defining the system of equations, while $P_{2n+2m}(0)=-a_0=(-1)^{n-1}\varepsilon$.

Solution of system (9) is carried out by the following algorithm.

Instead of a_0 we substitute $a_0=(-1)^n\varepsilon$ in all equations of system (9) and solve it as related to polynomial coefficients and ε which are received as functions of abscissa points of extremum x_q $(q=\overline{1,2n+1})$. By using the necessary condition for an extremum, and by taking into consideration that extremum is achieved at the points x_q $(q=\overline{1,2n+1})$, we obtain a system of equations for their determination. Solution of this system allows us to find the point of polynomial extremum and its coefficients and roots.

The above given algorithm allowed to find polynomials $P_4(x)(n=1, m=1)$, $P_6(x)(n=2, m=1)$, $P_8(x)(n=3, m=1)$, $P_{10}(x)(n=4, m=1)$:

$$P_4(x)=(x^2-1)(x-\sqrt{2}+1)(x+\sqrt{2}-1),$$

$$P_6(x)=(x^2-1)(x^4-0.607695x^2+0.038476),$$

$$P_8(x)=(x^2-1)(x^6\text{-}1.0791322598x^4+0.2717399132x^2\text{-}0.0091242781);$$

$$P_{10}(x)=(x^2-1)(x^8\text{-}1.5627140773x^6+0.7359121274x^4\text{-}0.1056195488x^2+$$
$$+0.0022107049). \qquad (10)$$

Here are the graphs of the constructed polynomials (Fig. 1).

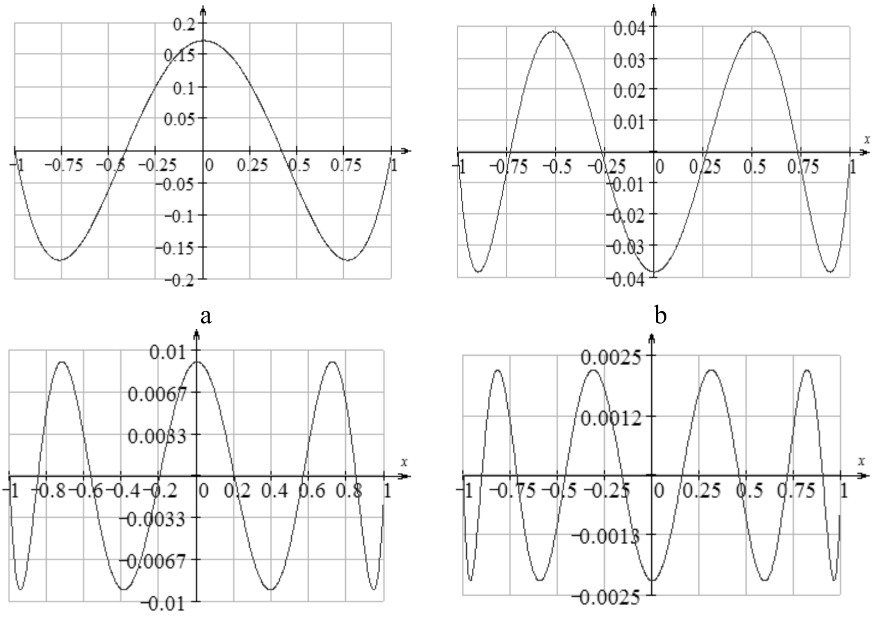

Fig.1. Graphs of polynomials $P_4(x)$ (a), $P_6(x)$ (b), $P_8(x)$ (c), $P_{10}(x)$ (d).

Table 1 shows the abscissas of extremum points, roots, belonging to the interval (–1,1), and norm in C[–1,1] for built in polynomials $P_{2n+2m}(x)$ $(n=1,2,3,4;\ m=1)$.

Table 1. The results of constructing polynomials by given method

Polynomial	Abscissas of extremum points	The roots of the polynomial belonging to (–1,1)	Norm $P_{2n+2m}(x)$ in $C[-1,1]$
$P_4(x)$	$-\sqrt{2-\sqrt{2}}$ 0.0000000000 $\sqrt{2-\sqrt{2}}$	$1-\sqrt{2}$ $\sqrt{2}-1$	$(\sqrt{2}-1)^2 \approx$ ≈ 0.171572875
$P_6(x)$	–0.8965754722 –0.5176380902 0.0000000000 0.5176380902 0.8965754722	–0.7320508076 –0.2679491923 0.2679491923 0.7320508076	0.0384757729
$P_8(x)$	–0.9419794124 –0.7209597940 –0.3901806083 0.0000000000 0.3901806083 0.7209597940 0.9419794124	–0.8477590650 –0.5664544974 –0.1989123674 0.1989123674 0.5664544974 0.8477590650	0.0091242781
$P_{10}(x)$	–0.9629115548 –0.8191014896 –0.5951120588 –0.3128689084 0.0000000000 0.3128689084 0.5951120588 0.8191014896 0.9629115548	–0.9021130328 –0.7159209562 –0.4596495484 –0.1583844401 0.1583844401 0.4596495484 0.7159209562 0.9021130328	0.0022107049

The use of constructed polynomials $P_{2n+2m}(x)$ for approximation of functions and solution of boundary value problems.

Here are the results of approximation of some functions that satisfy homogeneous boundary conditions at the points $x=\pm 1$ of interpolation polynomials, whose interpolation nodes are roots of polynomials constructed by us that the least deviate from zero. For comparison, we use approximation of the same as functions by Lagrange interpolation polynomials constructed on a uniform grid nodes.

Example 1. Interpolation function $y(x)=\dfrac{2e\cdot chx}{1+e^2}-1$ on the interval [–1,1]. Table 2 presents results of a calculation error of approximation of the given function by interpolation polynomial $\overline{L}_N y(x)$, which nodes are roots of polynomial $P_{2n+2m}(x)$ $(n=\overline{1,4};\ m=1)$ and satisfies the homogeneous boundary conditions at the points $x = \pm 1$ and the Lagrange interpolation polynomial $L_N y(x)$, constructed on a uniform grid nodes.

Table 2. The accuracy of the approximation of function (Example 1) by interpolation polynomials

Number of polynomial interpolation nodes	$\max\limits_{[-1,1]} \left\| y(x) - \bar{L}_N y(x) \right\|$	$\max\limits_{[-1,1]} \left\| y(x) - L_N y(x) \right\|$
4	$4.9072350446 \cdot 10^{-3}$	$5.6369735873 \cdot 10^{-3}$
6	$3.6124471569 \cdot 10^{-5}$	$6.4731786526 \cdot 10^{-5}$
8	$1.5143068530 \cdot 10^{-7}$	$4.7031213300 \cdot 10^{-7}$
10	$4.0453826399 \cdot 10^{-10}$	$2.2998706134 \cdot 10^{-9}$

Example 2. Interpolation of functions $y(x) = \cos\frac{\pi x}{2} + \mathrm{sh}(x^2) - \mathrm{sh}(1)$, $y(-1) = y(1) = 0$. Table 3 shows the results of a calculation error of approximation of the given function by $\bar{L}_N y(x)$ interpolation polynomials $\bar{L}_N y(x)$ and $L_N y(x)$.

Table 3. The accuracy of the approximation of function (Example 2) by interpolation polynomials

Number of polynomial interpolation nodes	$\max\limits_{[-1,1]} \left\| y(x) - \bar{L}_n y(x) \right\|$	$\max\limits_{[-1,1]} \left\| y(x) - L_n y(x) \right\|$
4	0.2286725852	0.2639401248
6	0.0148322719	0.0268538989
8	$5.3976055322 \cdot 10^{-4}$	$1.6963767845 \cdot 10^{-3}$
10	$1.2469099963 \cdot 10^{-5}$	$7.2217151397 \cdot 10^{-5}$

Here are the results of computational experiments conducted to determine the optimal choice of interpolation nodes in solving the boundary-value problem (4)–(5), which approximate solution was presented in the form (6).

Two methods of finding nodes of approximating polynomials are suggested. The first method is to find the unknown nodes and nodal parameters of the approximate solution on condition of the minimum of functional (7), corresponding to boundary value problem solved. In this case, it is essential to solve an essentially nonlinear system of equations related to C, X.

The second method consists in finding an approximate solution in the form (6), where the interpolation nodes are fixed and coincide with the roots of polynomials constructed (10), which least deviate from zero and satisfy the boundary conditions. So in this case we find only fixed C_k in solving Ritz system of linear algebraic equations. To illustrate the above let us consider the following boundary value problem

$$y'' - y = f(x), \tag{11}$$

$$y(-1) = 0, \ \ y(1) = 0. \tag{12}$$

We are looking for a solution by the first method in the form (6) of the unknown X_k and C_k to find the minimizing functional

$$J(X,C) = \int_{-1}^{1}\left[\left(\frac{d}{dx}L_n(x,X,C)\right)^2 + L_n^{\,2}(x,X,C) + 2f(x)L_n(x,X,C)\right]dx$$

for nodes X_k $\left(k = \overline{1,N-1}\right)$ and constant C_k $\left(k = \overline{1,N-1}\right)$.

Here are the numerical results with $f(x) = 1$ (Table 4)

Table 4. Results of computational experiments with $f(x) = 1$

Number of nodes	First method		Second method	
	X_k	$\max\limits_{[-1,1]}\lvert y(x) - L_n\rvert$	X_k	$\max\limits_{[-1,1]}\lvert y(x) - L_n\rvert$
4	± 1.000000 ± 0.086001	$5.197131 \cdot 10^{-3}$	± 1.000000 ± 0.414214	$5.197131 \cdot 10^{-3}$
6	± 1.000000 ± 0.759778 ± 0.204215	$2.823498 \cdot 10^{-4}$	± 1.000000 ± 0.732051 ± 0.267949	$4.355245 \cdot 10^{-5}$
8	± 1.000000 ± 0.878089 ± 0.510938 ± 0.143578	$1.233824 \cdot 10^{-4}$	± 1.000000 ± 0.847759 ± 0.566454 ± 0.198912	$1.991277 \cdot 10^{-7}$
10	± 1.000000 ± 0.934849 ± 0.702301 ± 0.357477 ± 0.112478	$5.484291 \cdot 10^{-5}$	± 1.000000 ± 0.902113 ± 0.715921 ± 0.459649 ± 0.158384	$1.661589 \cdot 10^{-8}$

Table 4 (first method) presents the results of finding nodes of approximate solution of problem (11) – (12) for different values n and uniform initial approximation nodes, as well as results of the calculation error of approximation of the solution.

Table 4 (second method) shows the components of polynomial interpolation and approximation error solution of the problem (11)– (12) for the case when the interpolation nodes coincide with the roots of polynomials (10).

Problem (11)–(12) were considered as an example of application of the constructed polynomials to the solution of boundary value problems with the right side

$$f(x)=2ch(x^2)+(4x^2-1)sh(x^2)-\left(\frac{\pi^2}{4}+1\right)\cos\frac{\pi x}{2}+sh(1),$$

whose exact solution has the form

$$y(x)=\cos\frac{\pi x}{2}+sh(x^2)-sh(1).$$

Figure 2 shows the graph of approximate solution $u_n(x)$ constructed by the suggested polynomials, and the graph of the exact solution $y(x)$, which illustrates the quality of approximation.

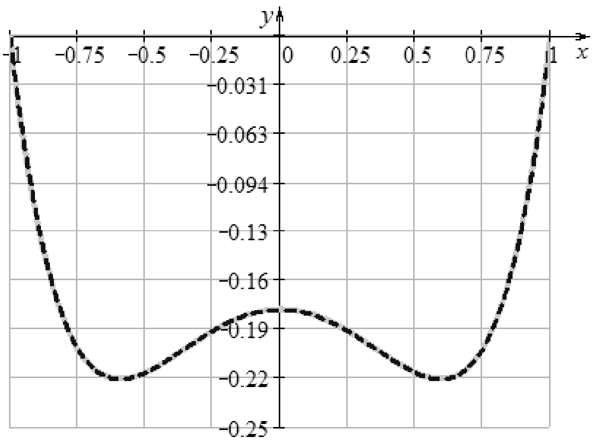

Fig. 2. Graph of approximate solution $u_n(x)$ (solid line) and graph of the exact solution $y(x)$ (dotted line)

Conclusions. The paper offers a method of constructing polynomials, which exactly satisfies the boundary conditions and at the same time the least deviates from zero. Examples of polynomials constructed by this method, and also examples of approximation of functions by interpolation polynomial with knots of interpolation which are the roots of the found polynomials. Examples confirm that the choice of interpolation nodes, which are zeros of polynomials satisfying the given boundary conditions and also the least deviate from zero on the interval $[-1, 1]$, gives a better approximation than the approximation resulting from interpolation of polynomials which exactly satisfy the boundary conditions and interpolate $y(x)$ on a uniform grid of nodes.

The results of numerical solution of boundary value problems for ordinary difference equations of 2nd order indicate that the use of polynomials with interpolation nodes that are roots of polynomials which deviate least from zero at the interval $[-1, 1]$ and satisfy the boundary conditions for the approximation demonstrates a better approximation accuracy than the approximation obtained by finding C, X on condition of t minimum of the functional $J(X, C)$, moreover, it doesn't demand finding the interpolation nodes.

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