THE FUNCTION $\mathrm{mup}_s(x)$ AND ITS APPLICATIONS TO THE THEORY OF GENERALIZED TAYLOR SERIES, APPROXIMATION THEORY AND WAVELET THEORY

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In this paper we consider compactly supported solutions of some functional differential equations and their applications.

Consider the equation

$$y'(x) = 2\cdot\sum_{k=1}^{s}\left(y(2s\cdot x + 2s - 2k + 1) - y(2s\cdot x - 2k + 1)\right), \qquad (1)$$

where $s = 1, 2, 3, \ldots$.

It was shown in [1] that the function $\mathrm{up}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\cdot\prod_{k=1}^{\infty}\frac{\sin(t\cdot 2^{-k})}{t\cdot 2^{-k}}dt$ is a solution with a compact support of the equation (1) in the case $s = 1$.

The function $\mathrm{up}(x)$ has applications in various branches of mathematics such as the theory of generalized Taylor series [2,3], approximation theory [3–6] and so on.

Let H_ρ be a class of functions $f \in C^{\infty}_{[-1,1]}$ such that $\left|f^{(n)}\right| \le c(f)\rho^n 2^{\frac{n(n+1)}{2}}$ for any $n = 0,1,2,\ldots$. It was proved in [3] that if $f(x)$ belongs to the class H_ρ, $\rho\in[1,2)$, then $f(x)$ expands in the generalized Taylor series

$$f(x) = \sum_{n=0}^{\infty}\sum_{k\in N_n} f^{(n)}\left(x_{n,k}\right)\cdot\phi_{n,k}(x),$$

where $N_0 = \{-1,0,1\}$ and $x_{0,k} = k$, $k\in N_0$; $N_n = \left\{-2^{n-1}, -2^{n-1}+1, \ldots, 2^{n-1}\right\}$ and $x_{n,k} = \dfrac{k}{2^{n-1}}$, $n \ne 0$, $k \in N_n$; basic functions $\phi_{n,k}(x)$ are defined by the following conditions: $\phi_{n,k}\in H_1$ and $\phi^{(m)}_{n,k}\left(x_{m,j}\right) = \delta^m_n\cdot\delta^j_k$ for $n, m = 0,1,2,\ldots$, $k\in N_n$, $j \in N_m$, δ^j_k is the Kronecker delta. The functions $\phi_{n,k}(x)$ are the finite linear combinations of translates of the function $\mathrm{up}(x)$:

$$\phi_{n,k}(x) = \sum_j c^{(n,k)}_j\cdot \mathrm{up}\left(x - \tfrac{j}{2^n}\right).$$

These functions are also similar to the functions x^n in common Taylor series.

In 1986 the list of unsolved problem about generalized Taylor series was stated in [7]. Some of these problems were solved by T.V. Rvachova [8–12].

We can see that the partial sum $S_m(x)=\sum_{n=0}^{m}\sum_{k\in N_n} f^{(n)}\left(x_{n,k}\right)\cdot\phi_{n,k}(x)$ of the generalized Taylor series is the finite linear combination of translates of the function $up(x)$: $S_m(x)=\sum_j d_{m,j}\cdot up\left(x-\frac{j}{2^n}\right)$. Therefore the problem of approximation properties of the space UP_n, which is a space of functions $\phi(x)$ such that $\phi(x)=\sum_j c_j\cdot up\left(x-\frac{j}{2^n}\right)$, $x\in[-1,1]$, is of interest.

Let $\overline{UP}_n$ be a space of functions $\phi(x)=\sum_j c_j\cdot up\left(\frac{x}{\pi}-\frac{j}{2^n}\right)$ such that $\phi^{(m)}(-\pi)=\phi^{(m)}(\pi)$ for any $m=0,1,2,\dots$. It was shown [5] that $\dim\overline{UP}_n=2^{n+1}$. Denote by $\tilde{W}_\infty^r$ a class of functions $f\in C_{[-\pi,\pi]}^r$ such that $f^{(k)}(-\pi)=f^{(k)}(\pi)$ for any $k=0,1,\dots,r-1$ and $\left\|f^{(r)}\right\|_{C[-\pi,\pi]}\le 1$. Let $\tilde{W}_2^r$ be a class of functions $f\in C_{[-\pi,\pi]}^{r-1}$ such that $f^{(k)}(-\pi)=f^{(k)}(\pi)$ for any $k=0,1,\dots,r-1$, $f^{(r-1)}(x)$ is absolutely continuous and $\left\|f^{(r)}\right\|_{L_2[-\pi,\pi]}\le 1$.

It was proved in [5] that $\lim_{n\to\infty}\frac{E_C\left(\tilde{W}_\infty^r,\overline{UP}_n\right)}{d_{2^{n+1}}\left(\tilde{W}_\infty^r,C\right)}=1$, where $E_X(A,L)=$ $=\sup_{\phi\in A}\inf_{\psi\in L}\|\phi-\psi\|_X$ is the best approximation of the class A by the space L in norm of the space X and $d_N(K,X)=\inf_{\dim L=N}\sup_{\phi\in K}\inf_{\psi\in L}\|\phi-\psi\|_X$ is the Kolmogorov width [13]. In other words, the spaces $\overline{UP}_n$ are asymptotically extremal for approximation of classes $\tilde{W}_\infty^r$ in norm of C.

It was shown in [3] that $E_{L_2}\left(\tilde{W}_2^r,\overline{UP}_n\right)=d_{2^{n+1}}\left(\tilde{W}_2^r,L_2\right)$ for any $n\ge n(r)$.This means that the spaces $\overline{UP}_n$ are asymptotically extremal for approximation of classes $\tilde{W}_2^r$ in norm of L_2.

Therefore the spaces of linear combinations of the function $up(x)$ translates have “good” approximation properties.

Let $s=2,3,4,\dots$. In this case the function $mup_s(x)=\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}e^{itx}\cdot F_s(t)dt$, where $F_s(t)=\prod_{k=1}^{\infty}\frac{\sin^2\left(\frac{st}{(2s)^k}\right)}{s^2\cdot\frac{t}{(2s)^k}\cdot\sin\left(\frac{t}{(2s)^k}\right)}$, is a solution with a compact support of the functional differential equation (1) [14].

Let $H_{\alpha,s}=\left\{f\in C^{\infty}_{[-1,1]}:\left|f^{(n)}(x)\right|\le c(f)\cdot\alpha^{n}\cdot 2^{n}\cdot(2s)^{\frac{n(n-1)}{2}},n=0,1,2,\dots\right\}$. The generalized Taylor series for these classes was introduced in [14]. It was proved in [14,15] that if $f\in H_{\alpha,s}$, $\alpha\in(1,2s)$, then $f(x)$ expands in the generalized Taylor series

$$f(x)=\sum_{n=0}^{\infty}\left(\sum_{k\in N_{s,n}}f^{(n)}\left(x_{s,n,k}\right)\cdot\phi_{s,n,k}(x)+\sum_{p\in D_{s,n}}\Delta^{2}_{\frac{1}{s(2s)^{n}}}\left(f^{(n)};x^{*}_{s,n,p}\right)\cdot\psi_{s,n,p}(x)\right),$$

where $\Delta^{2}_{h}(f;x)=f(x+h)-2\cdot f(x)+f(x-h)$; $N_{s,0}=\{-1,0,1\}$ and $x_{s,0,k}=k$ for $k\in N_{s,0}$; $N_{s,n}=\left\{-s(2s)^{n-1},\dots,s(2s)^{n-1}\right\}$ and $x_{s,n,k}=\frac{k}{s\cdot(2s)^{n-1}}$ for $k\in N_{s,n},n=1,2,\dots$; $D_{s,n}=\left\{1,2,\dots,(2s)^{n+1}\right\}\setminus\{k\cdot s\}$, $x^{*}_{s,n,p}=-1+\frac{p}{s\cdot(2s)^{n}}$ for $p\in D_{s,n},n=0,1,2,\dots$. The basic functions $\phi_{s,n,k}(x)$ and $\psi_{s,n,p}(x)$ are defined by the following conditions: $\phi_{s,n,k}\in H_{1,s},\psi_{s,n,p}\in H_{1,s}$, $\phi^{(m)}_{s,n,k}\left(x_{s,m,j}\right)=\delta^{m}_{n}\delta^{j}_{k}$, $\psi^{(m)}_{s,n,p}\left(x_{s,m,j}\right)=0$, $\Delta^{2}_{\frac{1}{s(2s)^{m}}}\left(\phi^{(m)}_{s,n,k};x^{*}_{s,m,q}\right)=0$, $\Delta^{2}_{\frac{1}{s(2s)^{m}}}\left(\psi^{(m)}_{s,n,p};x^{*}_{s,m,q}\right)=\delta^{m}_{n}\delta^{q}_{p}$ for any $n=0,1,2,\dots$, $m=0,1,2,\dots$, $k\in N_{s,n}$, $j\in N_{s,m}$, $p\in D_{s,n}$, $q\in D_{s,m}$.

The problem of the existence of the asymptotics of the basic functions $\phi_{s,n,k}(x)$, $\psi_{s,n,p}(x)$ as $n\to\infty$ was considered in [16–18]. We present some results obtained in these papers.

Let $\Phi_{s}(z)=\sum_{k=0}^{\infty}\mathrm{mup}_{s}\left(-1+\frac{1}{s(2s)^{k}}\right)z^{k}$, $\Lambda_{s}(z)=\sum_{k=0}^{\infty}\mathrm{mup}_{s}\left(-1+\frac{1}{(2s)^{k+1}}\right)\cdot z^{k}$ and $T_{s}(z)=\sum_{k=0}^{\infty}\mathrm{mup}_{s}\left(-1+\frac{1}{s\cdot(2s)^{k+1}}\right)\cdot z^{k}$.

Theorem 1. *For any* $s\ge 2$ *there exists the unique* $\lambda_{s}\in\left\{z\in C:|z|<4s^{2}\right\}$ *such that* $\Phi_{s}\left(z_{s}\right)=0$, *moreover* λ_{s} *is real and belongs to the interval* $(-3s^{2},-1)$ *and* $\Lambda_{s}\left(\lambda_{s}\right)\ne 0$, $\Lambda_{s}\left(\lambda_{s}\right)-s\cdot T_{s}(\lambda_{s})\ne 0$.

This fact was essentially used to prove the existence of the asymptotics of $\phi_{s,n,k}(x)$ and $\psi_{s,n,p}(x)$ as $n\to\infty$.

Consider the functions: $ab_{s}(x)=\sum_{j=0}^{\infty}\lambda^{j}_{s}\cdot\mathrm{mup}_{s}\left(x-1+\frac{1}{s\cdot(2s)^{j}}\right)$ for $x\le 0$ and

$$\Phi_{s,0}(x)=\begin{cases}ab_{s}(x),x\le 0\\ ab_{s}(-x),x>0\end{cases},\Phi_{s,1}(x)=\begin{cases}ab_{s}(x),x\le 0\\ -ab_{s}(-x),x>0\end{cases},\Psi_{s}(x)=\begin{cases}ab_{s}(x),x\le 0\\ 0,x>0\end{cases}.$$

Theorem 2. *For any* $s=2,3,4,\dots$ *and* $k=0,1,2,\dots$ *it is true that*

1) $\left\|\frac{d_{s}(2n)}{c_{s,2n}}\cdot\phi^{(k)}_{s,2n,0}(x)-\Phi^{(k)}_{s,0}(x)\right\|_{C[-1,1]}\le M_{1}\cdot\frac{(2n-k)\cdot\left|\lambda_{s}\right|^{2n}}{\rho^{2n}}$, $2n\ge k+5$;

2) $\left\|\frac{d_s(2n-1)}{c_{s,2n-1}}\cdot\phi^{(k)}_{s,2n-1,0}(x)-\Phi^{(k)}_{s,1}(x)\right\|_{C[-1,1]}\le M_2\,\frac{(2n-k-1)\left|\lambda_s\right|^{2n-1}}{\rho^{2n-1}},2n\ge k+6;$

3) $\left\|\frac{d_s(n)}{b_{s,n}}\cdot\psi^{(k)}_{s,n,s\cdot(2s)^n-1}(x)-\Psi^{(k)}_{s}(x)\right\|_{C[-1,1]}\le M_3\cdot\frac{(n-k)\cdot\left|\lambda_s\right|^{n}}{\rho^{n}},n\ge k+5;$

where $\rho\in\left[3s^2;4s^2\right)$, $d_s(n)=2^n\cdot(2s)^{\frac{n(n-1)}{2}}$, $b_{s,n}=-\frac{1}{\lambda_s^n}\cdot\mathrm{Res}_{\lambda_s}\left(\frac{\Lambda_s(z)-s\cdot T_s(z)}{\Phi_s(z)}\right)$ *and* $c_{s,n}=-\frac{1}{\lambda_s^n}\cdot\mathrm{Res}_{\lambda_s}\left(\frac{\Lambda_s(z)}{\Phi_s(z)}\right)$,

It follows from this theorem that for any $s=2,3,4,\dots$ and $k=0,1,2,\dots$ functions $\frac{d_s(2n)}{c_{s,2n}}\cdot\phi^{(k)}_{s,2n,0}(x)$, $\frac{d_s(2n-1)}{c_{s,2n-1}}\cdot\phi^{(k)}_{s,2n-1,0}(x)$ and $\frac{d_s(n)}{b_{s,n}}\cdot\psi^{(k)}_{s,n,s(2s)^n-1}(x)$ converge uniformly on $[-1,1]$ respectively to $\Phi^{(k)}_{s,0}(x)$, $\Phi^{(k)}_{s,1}(x)$ and $\Psi^{(k)}_{s}(x)$ as $n\to\infty$.

In this paper we consider application of the function $\mathrm{mup}_s(x)$ to the wavelet theory.

Note that the problem about approximation properties of spaces of finite linear combinations of translates of the function $\mathrm{mup}_s(x)$ is open.

Nonstationary compactly supported wavelets. Let $s=2^m$, where $m=1,2,3,\dots$. Then $F_{2^m}(t)=\prod_{k=1}^{\infty}\frac{\sin\left(\frac{2^m\cdot t}{2^{k(m+1)}}\right)}{\frac{2^m\cdot t}{2^{k(m+1)}}}\cdot\prod_{j=0}^{m-1}\cos\left(\frac{2^j\cdot t}{2^{k(m+1)}}\right)$.

Consider the following functions:

$$F_{2^m,n,0}(t)=\left(\frac{\sin\left(\frac{2^m\cdot t}{2^{(m+1)(n+1)}}\right)}{\frac{2^m\cdot t}{2^{(m+1)(n+1)}}}\right)^n\cdot F_{2^m}\left(\frac{t}{2^{n(m+1)}}\right),$$

$$F_{2^m,n,k}(t)=\left(\frac{\sin\left(\frac{2^{m-k}\cdot t}{2^{(m+1)(n+1)}}\right)}{\frac{2^{m-k}\cdot t}{2^{(m+1)(n+1)}}}\right)^{n+1}\cdot\prod_{j=0}^{m-1-k}\cos\left(\frac{2^j\cdot t}{2^{(m+1)(n+1)}}\right)\cdot F_{2^m}\left(\frac{t}{2^{(n+1)(m+1)}}\right),$$

where $n=0,1,2,\dots$ and $k=1,\dots,m$.

Let $\mathrm{Fmup}_{2^m,n,k}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{itx}\cdot F_{2^m,n,k}(t)dt$ for $n=0,1,2,\dots,k=0,1,\dots,m$.

The function $F_{2^m,n,k}(t)$ is an entire function of an exponential type. It follows from the Wiener–Paley theorem [19] that $\mathrm{Fmup}_{2^m,n,k}(x)=0$ for any x such that $|x|>\frac{n+2}{2^{n(m+1)+k+1}}$. Moreover, since $F_{2^m,n,k}(t)$ approaches to zero faster then t^j for any $j=1,2,3,\dots$, it follows that $\mathrm{Fmup}_{2^m,n,k}\in C^\infty$.

Notice that $\mathrm{Fmup}_{2^m,0,0}(x)=\mathrm{mup}_{2^m}(x)$.

Let $v_{2^m,n(m+1)+k}(x)=\mathrm{Fmup}_{2^m,n,k}\left(x-\frac{n+2}{2^{n(m+1)+k+1}}\right)$, where $n=0,1,2,\dots$ and $k=0,1,\dots,m$.

Denote by $V_{2^m,n}$ the space of functions $\phi(x)$ such that

$$\phi(x)=\sum_{j\in I(\phi)}c_j\cdot v_{2^m,n}\left(x-\frac{j}{2^n}\right),\quad x\in R,$$

where $I(\phi)$ is a finite subset of integers and $n=0,1,2,\dots$.

By construction, $F_{2^m,n,k}(t)=\left(\cos\left(\frac{2^{m-k-1}\cdot t}{2^{(m+1)(n+1)}}\right)\right)^{n+2}\cdot F_{2^m,n,k+1}(t)=$

$$=\frac{1}{2^{n+2}}\cdot e^{-i\cdot\frac{2^{m-k-1}(n+2)t}{2^{(m+1)(n+1)}}}\cdot\left(e^{i\cdot\frac{2^{m-k}t}{2^{(m+1)(n+1)}}}+1\right)^{n+2}\cdot F_{2^m,n,k+1}(t),$$

$$F_{2^m,n,m}(t)=\left(\cos\left(\frac{2^m\cdot t}{2^{(m+1)(n+2)}}\right)\right)^{n+1}\cdot F_{2^m,n+1,0}(t)=$$

$$=\frac{1}{2^{n+1}}\cdot e^{-i\cdot\frac{2^m(n+1)}{2^{(m+1)(n+2)}}}\cdot\left(e^{i\cdot\frac{2^{m+1}t}{2^{(m+1)(n+2)}}}+1\right)^{n+1}\cdot F_{2^m,n+1,0}(t)$$

for any $n=0,1,2,\dots$ and $k=0,1,\dots,m-1$. Combining this with properties of Fourier transform we obtain that

$$\mathrm{Fmup}_{2^m,n,k}(x)=\frac{1}{2^{n+2}}\cdot\sum_{j=0}^{n+2}\binom{n+2}{j}\cdot\mathrm{Fmup}_{2^m,n,k+1}\left(x+\frac{2^{m-k}\cdot j-2^{m-k-1}\cdot(n+2)}{2^{(m+1)(n+1)}}\right),\qquad(2)$$

$$\mathrm{Fmup}_{2^m,n,m}(x)=\frac{1}{2^{n+1}}\cdot\sum_{j=0}^{n+1}\binom{n+1}{j}\cdot\mathrm{Fmup}_{2^m,n+1,0}\left(x+\frac{2^{m+1}\cdot j-2^m\cdot(n+2)}{2^{(m+1)(n+2)}}\right).\qquad(3)$$

This implies that $V_{2^m,0}\subset V_{2^m,1}\subset\dots\subset V_{2^m,n}\subset\dots$.

Define the inner product of two functions $f,g\in L_2(R)$ as the integral $\int_R f(x)\cdot g(x)dx$.

Let $W_{2^m,n}=\left\{f\in V_{2^m,n}:f\perp V_{2^m,n-1}\right\}$ for any $n=1,2,3,\dots$.

Theorem 3. *For any* $m=1,2,3,\dots$ *and* $n=1,2,3,\dots$ *there exists the function* $\mathrm{wup}_{2^m,n}(x)$ *such that*

1) the system of functions $\left\{\mathrm{wup}_{2^m,n}\left(x-\frac{j}{2^{n-1}}\right)\right\}_{j\in Z}$ *is a basis of the linear space* $W_{2^m,n}$;

2) supp $\mathrm{wup}_{2^m,n}(x)\subseteq\left[0,\frac{n+2}{2^{n-1}}\right]$.

The linear space $W_{2^m,n}$ is a space of wavelets. The system of functions

$$\Omega=\left\{\mathrm{mup}_{2^m}(x-j),\mathrm{wup}_{2^m,n}\left(x-\frac{j}{2^{n-1}}\right)\right\}_{n\in N,j\in Z}$$

is a system of nonstationary wavelets with a compact support. Since spaces $V_{2^m,n}$ consist of finite linear combinations of translates of smooth functions, we see that for any $n=1,2,3,\dots$ the function $wup_{2^m,n}(x)$ is also smooth.

To prove theorem 3, we need several lemmas.

Lemma 1. *It is true that* $supp\, v_{2^m,\ell(m+1)+k} \subseteq \left[0;\frac{\ell+2}{2^{\ell(m+1)+k}}\right]$ *for any* $\ell=0,1,2,\dots$ and $k=0,1,\dots,m$.

The ***proof*** is trivial.

Lemma 2. *It is true that* $v_{2^m,\ell(m+1)+k}(x)>0$ *for any* $x\in\left(0;\frac{1}{2^{\ell(m+1)+k}}\right)\cup \cup\left(\frac{\ell+1}{2^{\ell(m+1)+k}};\frac{\ell+2}{2^{\ell(m+1)+k}}\right)$ *and* $\ell=0,1,2,\dots$, $k=0,1,\dots,m$.

Proof. To prove this statement, we need the following properties of the function $mup_{2^m}(x)$:

1) the function $mup_{2^m}(x)$ is even and $mup_{2^m}(0)=1$ [14];

2) the function $mup_{2^m}(x)$ increases on the segment [–1;0] [20];

3) $mup_{2^m}\left(-1+\frac{1}{2^{r(m+1)}}\right)=\frac{2^r}{(r-1)!\cdot 2^{\frac{(m+1)r(r+1)}{2}}}\cdot\int_0^1 x^{r-1}\cdot mup_{2^m}(x)dx$ for any $r=1,2,\dots$ [20].

These properties provides that $mup_{2^m}(x)>0$ for any $x\in(-1;1)$. Furthermore, it follows from (2) and (3) that there exists a constant $C>0$ such that $mup_{2^m}(x)=C\cdot v_{2^m,\ell(m+1)+k}(x+1)$ for $x\in\left(-1;-1+\frac{1}{2^{\ell(m+1)+k}}\right)$ and $mup_{2^m}(x)=C\cdot v_{2^m,\ell(m+1)+k}\left(x-1+\frac{\ell+2}{2^{\ell(m+1)+k}}\right)$ for $x\in\left(1-\frac{1}{2^{\ell(m+1)+k}};1\right)$. This completes the proof.

Proof. of **theorem 3**. Let $n=\ell(m+1)+k$, where $\ell=0,1,2,\dots$, $k=1,\dots,m$.

First note that if $\phi(x)\in W_{2^m,n}$, then $\phi\left(x-\frac{j}{2^{n-1}}\right)\in W_{2^m,n}$ for any integer j.

For any $\phi\in V_{2^m,n}\setminus\{0\}$ it is true that $\phi(x)=\sum_{j=m(\phi)}^{M(\phi)} c_j\cdot v_{2^m,n}\left(x-\frac{j}{2^n}\right)$, where $c_{m(\phi)}\neq 0$, $c_{M(\phi)}\neq 0$ and $m(\phi)\le M(\phi)$.

It is clear that $W_{2^m,n}=\{0\}\cup L_1\cup L_2$, where $L_r=\{\phi\in W_{2^m,n}\setminus\{0\}: m(\phi)=2\tau-r,\tau\in Z\}$ and $r\in\{1;2\}$.

First let us prove that L_1 is empty.

Let $L_1 \neq \varnothing$. Then there exists $\phi(x)$ such that $m(\phi) = 2\tau - 1$. It follows from $\phi \perp V_{2^m, n-1}$ that $\int\limits_R \phi(x) \cdot v_{2^m, n-1}\left(x - \frac{\tau - (\ell+2)}{2^{n-1}}\right)dx = 0$. On the other hand it follows from lemma 1 and lemma 2 that $\int\limits_R \varphi(x) \cdot v_{2^m, n-1}\left(x - \frac{\tau - (\ell+2)}{2^{n-1}}\right)dx =$

$$= c_{m(\phi)} \cdot \int\limits_{\frac{\tau}{2^{n-1}} - \frac{1}{2^n}}^{\frac{\tau}{2^{n-1}}} v_{2^m, n}\left(x - \frac{m(\phi)}{2^n}\right) \cdot v_{2^m, n-1}\left(x - \frac{\tau - (\ell+2)}{2^{n-1}}\right)dx \neq 0.$$ Thus $L_1 = \varnothing$.

Now let us prove that $L_2 \neq \varnothing$.

Consider the function $\phi(x) = \sum\limits_{j=0}^{3\ell+4} c_j \cdot v_{2^m, n}\left(x - \frac{j}{2^n}\right)$. Let us prove that there exists coefficients $\{c_j\}$ such that $\phi \perp V_{2^m, n-1}$. It follows from lemma 1 that the function $\phi(x)$ is orthogonal to $V_{2^m, n-1}$, if and only if $\{c_j\}_{j=0}^{3\ell+4}$ satisfy the following system of linear algebraic equations:

$$\sum_{j=0}^{3\ell+4} c_j \cdot \int\limits_R v_{2^m, n}\left(x - \frac{j}{2^n}\right) \cdot v_{2^m, n-1}\left(x - \frac{r}{2^{n-1}}\right)dx = 0, r = -\ell - 1, \ldots, 2\ell + 2.$$

It is obvious that this system has a solution. Therefore $L_2 \neq \varnothing$. Moreover there exists $\phi \in L_2$ such that $\operatorname{supp}\phi(x) \subseteq \left[0; \frac{2\ell+3}{2^{n-1}}\right] \subseteq \left[0; \frac{n+2}{2^{n-1}}\right]$.

So there exists $wup_{2^m, n}(x) \in L_2$ such that

$$M\left(wup_{2^m, n}\right) - m\left(wup_{2^m, n}\right) = \min_{\phi \in L_2}\left(M(\phi) - m(\phi)\right). \qquad (4)$$

Without loss of generality it can be assumed that $m\left(wup_{2^m, n}\right) = 0$. By the above $wup_{2^m, n}(x) \subseteq \left[0; \frac{n+2}{2^{n-1}}\right]$.

It remains to check that the system of functions $\left\{wup_{2^m, n}\left(x - \frac{j}{2^{n-1}}\right)\right\}_{j \in Z}$ is a basis of the linear space $W_{2^m, n}$.

First it follows from lemma 2 that if $\sum\limits_{j \in I} \lambda_j \cdot wup_{2^m, n}\left(x - \frac{j}{2^{n-1}}\right) \equiv 0$, where I is a finite set of integers, then $\lambda_j = 0$ for any $j \in I$.

Let $I = \bigcup\limits_{\phi \in W_{2^m, n} \setminus \{0\}} \{M(\phi) - m(\phi)\}$. It is true that $I = \bigcup\limits_{j=0}^{\infty} \{d_j\}$, where $d_j < d_{j+1}$ for any $j = 0, 1, 2, \ldots$.

Consider any function $\phi \in W_{2^m,n} \setminus \{0\}$ such that $M(\phi) - m(\phi) = d_0$. We get $d_0 = \min I$. Combining this with (4), we obtain that $\phi(x) = \alpha \cdot \mathrm{wup}_{2^m,n}\left(x - \frac{\tau}{2^{n-1}}\right)$, where $\tau = \frac{m(\phi)}{2} \in Z$.

Let $r = 1, 2, 3, \ldots$.

Suppose that for any $j = 0, 1, \ldots, r$ and any function $\phi \in W_{2^m,n} \setminus \{0\}$ such that $M(\phi) - m(\phi) = d_j$ the function $\phi(x)$ is a finite linear combination of functions $\left\{\mathrm{wup}_{2^m,n}\left(x - \frac{j}{2^{n-1}}\right)\right\}_{j \in Z}$.

Consider any function $\psi \in W_{2^m,n} \setminus \{0\}$ such that $M(\psi) - m(\psi) = d_{r+1}$. We get $\psi(x) = \sum_{i=m(\psi)}^{M(\psi)} x_i \cdot v_{2^m,n}\left(x - \frac{i}{2^n}\right)$ and $\mathrm{wup}_{2^m,n}(x) = \sum_{i=0}^{M_0} c_i \cdot v_{2^m,n}\left(x - \frac{i}{2^n}\right)$, where $M_0 = M\left(\mathrm{wup}_{2^m,n}\right)$.

Let $\xi(x) = \psi(x) - \frac{x_{m(\psi)}}{c_0} \cdot \mathrm{wup}_{2^m,n}\left(x - \frac{\tau}{2^{n-1}}\right)$, where $\tau = \frac{m(\psi)}{2} \in Z$. Note that $M(\xi) - m(\xi) < d_{r+1}$. By the inductive assumption, the function $\xi(x)$ is a finite linear combination of functions $\left\{\mathrm{wup}_{2^m,n}\left(x - \frac{j}{2^{n-1}}\right)\right\}_{j \in Z}$. Hence there exists finite set of coefficients $\{f_i\}$ such that $\psi(x) = \sum_i f_i \cdot \mathrm{wup}_{2^m,n}\left(x - \frac{i}{2^{n-1}}\right)$. This completes the proof for the case $n = \ell(m+1) + k$, where $\ell = 0, 1, 2, \ldots$, $k = 1, \ldots, m$.

By the same argument we can prove this theorem for the case $n = \ell(m+1)$, where $\ell = 1, 2, 3, \ldots$.

Theorem 3 is a generalization of results obtained in [21] for the case $m = 1$.

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