APPROXIMATION OF ANALYTIC FUNCTIONS BY r–REPEATED DE LA VALLEE POUSSIN SUMS

[1]Novikov O. O., [2]Rovenska O. G.

[1]Slovyansk State Pedagogical University, Slovyansk, Ukraine
[2]Donbass State Machinebuilding Academy, Kramatorsk, Ukraine

The work concerns the questions of approximation of periodic (ψ,β)–differentiable functions of high smoothness by repeated arithmetic means of Fourier sums. One of the classifications of periodic functions nowadays is the classification suggested by A. Stepanets [1] which is based on the concept of (ψ,β)–differentiation. The given classification allows distinguishing all classes of summable periodic functions from the functions where the Fourier series can deviate to infinitely differentiable functions including analytical and entire ones. When choosing the parameters properly, classes of (ψ,β)–differentiable functions exactly coincide with the well-known classes of Vail differentiable functions, Sobolev classes W_p^l and classes of convolutions with integral kernels.

1. Problem formulation. Sets of (ψ,β)–differentiable functions are defined in the following way [1, p. 25–32].

Let f be a summable, 2π–periodic function and

$$S[f]=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=0}^{\infty}A_k(f;x)$$

be its Fourier series. Let $\psi(k)$ be an arbitrary numerical sequence and $\beta\in R$. Then, if the series

$$\sum_{k=1}^{\infty}\frac{1}{\psi(k)}(a_k\cos(kx+\frac{\beta\pi}{2})+b_k\sin(kx+\frac{\beta\pi}{2}))$$

is Fourier series of some summable function, this function is called (ψ,β)–derivative of function f and is denoted by f_β^ψ. The set of functions having (ψ,β)–derivative is denoted by L_β^ψ, and the subset of continuous functions from L_β^ψ is denoted by C_β^ψ. Besides, if (ψ,β)–derivative is almost everywhere bounded by unity, the set of such functions is denoted by $C_{\beta,\infty}^\psi$.

Numerical sequence $\psi(k)$, giving the class is possible to select only from the set of all positive convex downwards and disappearing on the infinity sequences. In this case approximable properties of classes C_β^ψ are characterized by the rate of functions $\psi(k)$ tending to zero.

We consider the case when sequence $\psi(k)$ is defined by relationship $\psi(k)=q^k,\quad q\in(0;1)$. Then the classes $C_{\beta,\infty}^\psi$ consist of analytical functions which can be regularly extended in the corresponding strip $|\operatorname{Im} z| < \ln \frac{1}{q}$. In this case classes $C^{\psi}_{\beta,\infty}$ are denoted by $C^{q}_{\beta,\infty}$. The functions from classes $C^{q}_{\beta,\infty}$ in each point can be presented in the form of Poisson integrals with an accuracy to the constant component

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{\psi}_{\beta}(x+t)\sum_{k=1}^{\infty} q^k \cos(kt + \frac{\beta\pi}{2})dt.$$

Let $\Lambda = \| \lambda_k^{(n)} \|$, $k, n = 1, 2, \dots$ is an infinite numerical matrix $\lambda_k^{(n)} = 0$, $k \geq n$. Each matrix of such a kind, on the basis of the Fourier series, gives a certain sequence of linear polynomial operators

$$U_n(f;x;\Lambda) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx).$$

For arbitrary natural $p < n$ the polynomials that are given by relationship

$$\lambda_k^{(n)} = \begin{cases} 1, & 1 \leq k \leq n-p-1, \\ 1 - \dfrac{k-n+p}{p}, & n-p \leq k \leq n-1 \end{cases}$$

are called de la Vallee Poussin sums. De la Vallee Poussin sums are also arithmetic means of the last p Fourier sums

$$V_{n,p}(f;x) = \frac{1}{p}\sum_{k=n-p}^{n-1} S_k(f;x).$$

Let p_1, $p_2 \in N$, $p_1 + p_2 < n$. The polynomials

$$V^{(2)}_{n,p}(f;x) = \frac{1}{p_1}\sum_{k=n-p_1}^{n-1} V_{k+1,p_2}(f;x) = \frac{1}{p_1 p_2}\sum_{k=n-p_1}^{n-1}\sum_{m=k-p_2+1}^{k} S_m(f;x)$$

will be called repeated de la Vallee Poussin sums [2]. If $p_1 = 1$ or $p_2 = 1$ these polynomials are de la Vallee Poussin sums, if $p_1 = p_2 = 1$, they are Fourier sums.

For arbitrary natural p_1, $p_2, \dots$, p_r and $\Sigma_p = \sum_{i=1}^{r} p_i < n$ the polynomials that are given by relationship

$$V^{(r)}_{n,\bar{p}}(f;x) = \frac{1}{p_1}\sum_{k_1=n-p_1}^{n-1} \frac{1}{p_2}\sum_{k_2=k_1-p_2+1}^{k_1} \dots \frac{1}{p_r}\sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}} S_{k_r}(f;x)$$

are called r –repeated de la Vallee Poussin sums.

We shall consider quantities $\delta_n(f;x;\Lambda) = f(x) - U_n(f;x;\Lambda)$ which are deviations of polynomials $U_n(f;x;\Lambda)$ from functions $f(x) \in C^{q}_{\beta,\infty}$.

The problem concerning the study of asymptotic behavior of quantities

$$\varepsilon(F;U_n)=\sup_{f\in F}\|f(x)-U_n(f;x;\Lambda)\|_C,\ n\to\infty$$

according to A. Stepanets is called the Kolmogorov–Nikol'skiy problem. If function $\varphi(n)$ is bound in an explicit form and the following relationship is performed

$$\varepsilon(F;U_n)=\varphi(n)+o(\varphi(n)),$$

then the Kolmogorov–Nikol'skiy problem is solved for the class of functions F and method U_n.

For upper bounds of deviations of Fourier sums on the classes of analytical functions S. Nikol'skiy [3] obtained the asymptotic equality:

$$\varepsilon(C^q_{\beta,\infty};S_n)=\sup_{f\in C^q_{\beta,\infty}}\|f(x)-S_n(f;x)\|_C=\frac{8q^n}{\pi^2}K(q)+O(1)\frac{q^n}{n},\ n\to\infty,$$

where

$$K(q)=\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-q^2\sin^2u}}$$

is the complete elliptical integral of the first kind, $O(1)$ is quantity uniformly bounded with respect to n. In [4] S. Stechkin proposed another proof of this result which made it possible to refine the remainder in the formula:

$$\varepsilon(C^q_{\beta,\infty};S_n)=\frac{8q^n}{\pi^2}K(q)+O(1)\frac{q^n}{n(1-q)},$$

where $O(1)$ is quantity uniformly bounded with respect to n, q.

Asymptotic equalities for upper bounds of the deviations of de la Vallee Poussin sums on the classes $C^q_{\beta,\infty}$ may be found in [5] (look also [6, 7, 8]):

$$\varepsilon(C^q_{\beta,\infty};V_{n,p})=\sup_{f\in C^q_{\beta,\infty}}\|f(x)-V_{n,p}(f;x)\|_C=$$

$$=\frac{4q^{n-p+1}}{\pi p(1-q^2)}+O(1)\left(\frac{q^{n-p+1}}{p(n-p)(1-q)^3}+\frac{q^n}{p(1-q)^2}\right),$$

where $O(1)$ is quantity uniformly bounded with respect to n, p and q.

2. Main Results. For upper bounds of deviations of the r–repeated de la Vallee Poussin sums on the classes of analytical functions $C^q_{\beta,\infty}$ the following statement was obtained.

Theorem. *Suppose that* $q \in (0;1)$, $\beta \in R$ *and* $p_i = p_i(n)$, $i = 1,2,\dots,r$ *are arbitrary natural numbers,* $\Sigma_p = \sum_{i=1}^{r} p_i < n$. *Then the following relations hold as* $n \to \infty$, $n - \Sigma_p \to \infty$

$$\varepsilon(C^q_{\beta,\infty}; V^{(r)}_{n,\bar p}) = \frac{4q^{n-\Sigma_p+r-1}}{\pi^2 \prod_{i=1}^{r} p_i} \int_0^{\pi} Z_q^{r+1}(x)dx + O(1)\prod_{i=1}^{r}\frac{1}{p_i} \times$$

$$\times\Big(\frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r-1)(1-q)^{r+2}} + \sum_{\alpha_{r-1}\subset \bar r} \frac{q^{n-1-\sum_{j\in\alpha_{r-1}} p_j + r}}{(1-q)^{r+1}}\Big), \qquad (1)$$

where $Z_q(x) = (1-2q\cos x + q^2)^{-1/2}$, α_{r-1} *is an arbitrary* $r-1$*–element's subset of set* $\bar r = \{1,2,\dots,r\}$, $|\alpha_r|$ *is number of elements of set* α_r, $O(1)$ *is quantity uniformly bounded with respect to* n, q, β, p_i, $i = 1,2,\dots,r$.

Proof. The statement of the theorem is proved using the procedure proposed by A. Stepanets in [1, p. 123–127].

While constructing the proof, first the convenient integral representations for quantities $\delta_n(f;x;V^{(r)}_{n,\bar p})$ were found

$$\delta_n(f;x;V^{(r)}_{n,\bar p}) = f(x) - \frac{1}{p_1}\sum_{k_1=n-p_1}^{n-1}\frac{1}{p_2}\sum_{k_2=k_1-p_2+1}^{k_1}\dots\frac{1}{p_r}\sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}} S_{k_r}(f;x) =$$

$$= \frac{1}{\prod_{i=1}^{r} p_i}\sum_{k_1=n-p_1}^{n-1}\sum_{k_2=k_1-p_2+1}^{k_1}\dots\sum_{k_r=k_{r-1}-p_r+1}^{k_{r-1}}(f(x) - S_{k_r}(f;x)) =$$

$$= \frac{1}{\pi\prod_{i=1}^{r} p_i}\int_{-\pi}^{\pi}\frac{f^q_\beta(x+t)}{(1-2q\cos t + q^2)^{r+1}}\sum_{\alpha\subset\bar r}(-1)^{(r-|\alpha|)} \times \qquad (2)$$

$$\times\sum_{\nu=0}^{r+1}(-1)^\nu C^\nu_{r+1} q^{(n-1-\sum_{j\in\alpha} p_j + r + \nu)}\cos[(n-1-\sum_{j\in\alpha} p_j + r - \nu)t + \frac{\beta\pi}{2}]dt,$$

where $|\alpha|$ is number of elements of set α, $\bar r = \{1,2,\dots,r\}$.

Further let

$$b^\beta_m(t) = (1-2q\cos t + q^2)^{-(r+1)/2}\cos(mt + \frac{\beta\pi}{2} + (r+1)\frac{q\sin t}{1-q\cos t}).$$

Then the quantity $\delta_n(f;x;V^{(r)}_{n,\bar p})$ may be represented as follows

$$\delta_n(f;x;V^{(r)}_{n,\bar{p}})=\frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x+t)b^{\beta}_{n-\Sigma_p+r-1}(t)dt+$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\subset\bar{r}}q^{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x+t)b^{\beta}_{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}(t)dt. \qquad (3)$$

Taking into account that $f(x)\in C^{q}_{\beta,\infty}$, also (3) one can find the upper value for quantity $\varepsilon(C^{q}_{\beta,\infty};V^{(r)}_{n,\bar{p}})$

$$\varepsilon(C^{q}_{\beta,\infty};V^{(r)}_{n,\bar{p}})\le\frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}|b^{\beta}_{n-\Sigma_p+r-1}(t)|\,dt+$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\subset\bar{r}}q^{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}\int_{-\pi}^{\pi}|b^{\beta}_{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}(t)|\,dt. \qquad (4)$$

Using work [1, p. 124] (also [5, p. 235–238]), further we find function $f_0(x)\in C^{q}_{\beta,\infty}$ for which this value cannot be improved. As

$$\int_{-\pi}^{\pi}|b^{\beta}_{m}(t)|\,dt=O(1)\int_{-\pi}^{\pi}\frac{dt}{\left(\sqrt{1-2q\cos t+q^2}\right)^{r+1}}=O(1)\frac{1}{(1-q)^{r+1}},$$

the quantity $\delta_n(f;x;V^{(r)}_{n,\bar{p}})$ can be rewritten as follows

$$\delta_n(f;x;V^{(r)}_{n,\bar{p}})=\frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x+t)b^{\beta}_{n-\Sigma_p+r-1}(t)dt+$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\subset\bar{r}}\frac{q^{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}. \qquad (5)$$

Based on (5), for any function $f\in C^{q}_{\beta,\infty}$ the following equality is true

$$\delta_n(f;0;V^{(r)}_{n,\bar{p}})=\frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r}p_i}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x)b^{\beta}_{n-\Sigma_p+r-1}(t)dt+$$

$$+O(1)\frac{1}{\prod_{i=1}^{r}p_i}\sum_{\alpha_{r-1}\subset\bar{r}}\frac{q^{n-1-\sum_{j\in\alpha_{r-1}}p_j+r}}{(1-q)^{r+1}}. \qquad (6)$$

Functions $b^{\beta}_{n-\Sigma_p+r-1}(t)$ may be refined on the set, the measure of which is less than $K(n-\Sigma_p+r-1)^{-1}q(1-q)^{-1}$, so that the following condition for new functions $b^{\beta,1}_{n-\Sigma_p+r-1}(t)$ will be fulfilled [5, p. 235–238]

$$\int_{-\pi}^{\pi} \text{sign}\ b^{\beta,1}_{n-\Sigma_p+r-1}(t)dt = 0.$$

Then the function $\text{sign}\ b^{\beta,1}_{n-\Sigma_p+r-1}(t)$ can be selected as a desire function $f_0(x)$ and next relation will hold

$$\int_{-\pi}^{\pi} f_0(t)b^{\beta}_{n-\Sigma_p+r-1}(t)dt = \int_{-\pi}^{\pi} |\, b^{\beta}_{n-\Sigma_p+r-1}(t)\,|\, dt + \frac{O(1)q}{(n-\Sigma_p+r-1)(1-q)^{r+2}}. \qquad (7)$$

In doing so for the found function the equality is true

$$\delta_n(f_0;0;V^{(r)}_{n,\bar{p}}) = \frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r} p_i}\int_{-\pi}^{\pi} |\, b^{\beta}_{n-\Sigma_p+r-1}(t)\,|\, dt +$$

$$+O(1)\frac{1}{\prod_{i=1}^{r} p_i}\Big(\sum_{\alpha_{r-1}\subset\bar{r}} \frac{q^{n-1-\sum_{j\in\alpha_{r-1}} p_j+r}}{(1-q)^{r+1}} + \frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r-1)(1-q)^{r+2}}\Big). \qquad (8)$$

Comparing relationships (4) and (8) we get asymptotic formula

$$\varepsilon(C^q_{\beta,\infty};V^{(r)}_{n,p}) = \sup_{f\in C^q_{\beta,\infty}} \|\, f(x) - V^{(r)}_{n,\bar{p}}(f;x)\,\|_C = \frac{q^{n-\Sigma_p+r-1}}{\pi\prod_{i=1}^{r} p_i}\int_{-\pi}^{\pi} |\, b^{\beta}_{n-\Sigma_p+r-1}(t)\,|\, dt +$$

$$+O(1)\frac{1}{\prod_{i=1}^{r} p_i}\Big(\sum_{\alpha_{r-1}\subset\bar{r}} \frac{q^{n-1-\sum_{j\in\alpha_{r-1}} p_j+r}}{(1-q)^{r+1}} + \frac{q^{n-\Sigma_p+r}}{(n-\Sigma_p+r-1)(1-q)^{r+2}}\Big). \qquad (9)$$

According to [5, p. 239–241] and counting the integral in the first component

$$\int_{-\pi}^{\pi} |\, b^{\beta}_{n-\Sigma_p+r-1}(t)\,|\, dt = \int_{-\pi}^{\pi} (1-2q\cos t + q^2)^{-(r+1)/2}\cos((n-\Sigma_p+r-1)t +$$

$$+\frac{\beta\pi}{2} + (r+1)\frac{q\sin t}{1-q\cos t})dt = \frac{4}{\pi}\int_{0}^{\pi} Z^{r+1}_q(x)dx + O(\frac{1}{n-\Sigma_p+r-1}),$$

we obtain the equality (1). The theorem is proved.

3. Conclusions. The asymptotic equality (1) provides the solution of the corresponding Kolmogorov–Nikol’skiy problem for the classes of analytic functions and the r –repeated de la Vallee Poussin sums in cases $p_i \to \infty$, $n - \Sigma_p \to \infty$, $i = 1, 2, ..., r$.

REFERENCES

1. Stepanec A.I. Classification and approximation of periodic functions.– K., 1987. – 268 p. (in Russian).
2. Rovenska O. Approximation of classes of analytic functions by repeated arithmetic means of Fourier sums. // Computational Methods and Function Theory. Abstracts of Intern. Conf. – 2009. – P. 51.
3. Nikol’skiy S.M. Approximation of the functions by trigonometric polynomials in the mean. // Izv. Acad. Nauk. SSSR, Ser. Mathem. – 1946. – v. 10, N3. – P. 207–256. (in Russian).
4. Stechkin S.B. Estimation of the remainder of Fourier series for the differentiable functions. // Tr. Mat. Inst. Acad. Nauk SSSR. – 1980. – v. 145. – P. 126–151. (in Russian).
5. Rukasov V.I., Novikov O.A. Approximation of analytical functions by de la Vallee Poussin sums. // Tr. Inst. Mat. – 1998. – v. 20. – P. 228–241. (in Russian).
6. Rukasov V.I., Chaichenko S.O. Approximation of the classes of analytical functions by de la Vallee–Poussin sums. // Ukrainian Math. J. – 2003. –v. 55, N6. – P. 575–590.
7. Serdyuk A.S. Approximation of Poisson integrals by de la Vallee Poussin sums. // Ukrainian Math. J. – 2004. – v. 56, N 1. – P. 97–107.
8. Rovenska O.G., Novikov O.A. Approximation of Poisson integrals by repeated de la Vallee Poussin sums. // Nonlinear Oscillations. – 2010. – v. 13, N 1. – P. 108–111.