# EVALUATION OF SPECTRUMS OF INTEGRAL OPERATORS IN SPACES OF FUNCTIONS OF SEVERAL VARIABLES AND ITS APPLICATION

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Relationship between the smoothness of the kernel of the integral Fredholm operator $K : L^2(a,b) \to L^2(a,b)$ acting in the space of functions of one variable, and its spectrum has been studied in several papers (a detailed bibliography in [5], [6], [1]). As noted in [2], the results of the operators belong to certain norms, ideals derive theorems on the behavior of the Fourier coefficients and Fourier–Walsh series of functions, in particular, the nuclear operator with difference kernel $K(t,s) = f(t-s)$ implies the absolute convergence of Fourier series of a periodic function f In [7] we obtain upper bounds for the singular numbers of integral operators by $p-k$ –variations of functions introduced $p=1$ in [12], [10] and used in [10], [11] o analyze the behavior of the Fourier coefficients, for arbitrary $p$ concept $\mathrm{p-k}$ – variation, introduced in [7], it is convenient to describe (with a modulus of continuity function, or independently of it) of the order of convergence to zero singular values of the operator.

In this paper we obtain estimates of the order of decrease in non-increasing ordered singular values $s_N \equiv s_N(K)$ of the operator $K : L^2(Q^m) \to L^2(Q^n)$, acting by the formula $(\mathrm{Kf})(\mathrm{x}) = \int\limits_{\mathrm{Q^m}} \mathrm{K(x,y)f(y)dy}$ where $x \in \mathbb{R}^n$, $\mathrm{x} \equiv (\mathrm{x}_1,...,\mathrm{x}_\mathrm{n})$, $\mathrm{y} \in \mathbb{R}^\mathrm{m}$, $y \equiv (y_1,...,y_m)$, the function $K(x,y) \equiv K(x_1,...,x_n;y_1,...,y_m)$ is defined on a rectangular parallelepiped

$$Q^{n,m} \equiv Q^n \times Q^m \equiv \left\{ x \in \mathbb{R}^n,\, y \in \mathbb{R}^m \middle| a_j \le x_j \le b_j,\, j = \overline{1,n};\, c_i \le y_i \le d_i,\, i = \overline{1,m} \right\},$$

$\{s_N\}$ – sequence of singular numbers of $K$ .

It is well known that for functions of several variables, addresses a number of different definitions of variation (Arzela, Pierpont, Hardy, Vitali, Frechet, Tunnels, a set of variations Kronrod–Vitushkin), this is due to the fact that the important properties of functions of bounded variation of one variable in the multivariate case "split" – the classes of functions with different specific disabilities multidimensional variations have different properties (represented as the difference decreasing functions, the convergence of Fourier series, rectifiability graphics functions, etc., an overview of issues related to some of these definitions, see [3]). In this note we show that to investigate the behavior of the sequence $\{s_N\}$ should be used $p-k$ –analog) of Pierpont variation.

**Definition 1**. Let $\Delta(k^d)$ – a partition of a $d$–dimensional parallelepiped

$$Q^d \equiv \left\{ t \in \mathbb{R}^d \,\middle|\, a_j \le t_j \le b_j,\ j = \overline{1,d} \right\} \qquad (1)$$

on $k^d$ non-overlapping parallelepipeds, $k \in \mathbb{N}$ $\Omega_{k_1 \ldots k_d}$. For the function $f : Q^d \to \mathbb{R}$ set $\bigvee_{Q^d} p\left( f;\Delta(k^d)\right) = \sum_{k_1=0}^{k-1} \ldots \sum_{k_d=0}^{k-1} \omega^p_{k_1 \ldots k_d}$, where $\omega^p_{k_1 \ldots k_d}$ – he vibration function $f$ on $\Omega_{k_1 \ldots k_d}$, $p \ge 0$; $p-k$–variation of the function $f$ on $Q^d$ is the quantity $\bigvee_{Q^d} p(f;k) = \sup_{\Delta(k^d)} \bigvee_{Q^d} p\left( f;\Delta(k^d)\right)$, where the supremum is taken over all partitions $\Delta(k^d)$ of a fixed $k \in \mathbb{N}$.

Note that when $d=1$, $p=1$ the get-variation [12], [10] (in the terminology of [10] – module changes), with $d=1$, $p \ge 0$ – $p-k$–variations [7].

**Definition 2.** $p$–variation of the function $f : Q^d \to \mathbb{R}$ is defined to be

$$\bigvee_{Q^d} p(f) = \sup_{k \in \mathbb{N}} \bigvee_{Q^d} p(f;k).$$

Note that $p=1$ this definition gives a multi-dimensional variation of the Pierpont [13], $d=1$ and at random $p \ge 0$ – $p$–he Wiener–variation, many results concerning the function of bounded $p$–variation (in $d=1$), see [4].

**Definition 3**. Suppose that the function of two vector arguments $f(x,y)$ given by $Q^{n,m}$, $\bigvee^{(2)}_{Q^m} p\left( f(x,\cdot);k\right)$ – its $p-k$–variation in the second variable for fixed first, $p-k$–variation of the second variable of the function $f$ is the quantity

$$\bigvee^{(2)}_{Q^{n,m}} p(f;k) = \sup_{x \in Q^n} \bigvee^{(2)}_{Q^m} p\left( f(x,\cdot);k\right).$$

The modulus of continuity of functions $f : Q^{n,m} \to \mathbb{R}$ of two vector arguments in the second variable is defined, as usual, by the equation

$$\omega(f,h) = \sup_{x \in Q^n} \sup_{\substack{y_1, y_2 \in Q^m \\ \|y_1 - y_2\| \le h}} \left| f(x,y_1) - f(x,y_2) \right|.$$

**Theorem 1** (this result without proof is in [9]). *For any* $r \le 2$ *in* $k \to \infty$

$$s_{k^m}(K) = O\left( k^{-m} \left( \omega\left( K(\cdot,\cdot), \frac{\sqrt{m}}{k} \right) \right)^{\frac{r}{2}} \left( \underset{Q^{n,m}}{\mathrm{V}}{}^{(2)}_{2-r} \left( K(\cdot,\cdot); k \right) \right)^{\frac{1}{2}} \right).$$

Note that if $r > 2$ the variation is not defined. In the case of $m = n = 1$ (scalar case) from the above theorem implies the result [7] formulated there with all the consequences. We note a well-known multi-dimensional result that follows from this theorem.

**Corollary 1.** ([6], Chapter XI, § 10). *If a function* $K(x, y) \in L^2(Q^{n,m})$, $x \in Q^n$, $y \in Q^m$, *satisfies* $Lip\,\alpha$ *he second variable, then* $s_{k^m}(K) = O\left(k^{-(m/2+\alpha)}\right)$ *or* $s_N(K) = O\left(N^{-(1/2+\alpha/m)}\right)$.

***Proof of Corollary 1.*** In Theorem 1 let $r = 2$. Given that

$$\left( \underset{Q^{n,m}}{\mathrm{V}}{}^{(2)}_{0} \left( K(.,.); k \right) \right)^{\frac{1}{2}} = k^{\frac{m}{2}},$$

one can obtain the desired result.

This shows that even the case $r = 2$, that is $0 - k$ –variation, is meaningful – the factor on the right side, associated with variation, has in this order $k^{m/2}$, and, consequently, the coefficient of the module $\omega$ – order $k^{-m/2}$.

***Proof of Theorem 1.*** Let $\chi_{k_1,\dots,k_m}$ – the indicator function of the hypercube $\Omega_{k_1,\dots,k_m}$. Each function $\chi_{k_1,\dots,k_m}$ generates a linear functional $(f; \chi_{k_1,\dots,k_m}) = \int\limits_{\Omega_{k_1,\dots,k_m}} f(x,y)dy$ on $L^2(Q^m)$. Consider the subspace $\mathcal{H} \subset L^2(Q^{n,m})$ of functions $f$ of two vector arguments $x$, $y$, $x = \left\{x_i\right\}_{i=1}^n$, $y = \left\{y_i\right\}_{i=1}^m$, satisfying $k^m$ linear conditions

$$(f; \chi_{k_1,\dots,k_m}) = \int\limits_{\Omega_{k_1,\dots,k_m}} f(x,y)dy = 0,\ k_i = \overline{0, k-1},\ i = \overline{1,m}.$$

We believe that $K_{k_1,\dots,k_m}(x,y) = K(x, y_{k_1,\dots,k_m})$, where $y_{k_1,\dots,k_m} = \left\{c_i = a + k^{-1}(d_i - c_i)k_i\right\}_{i=1}^m$. Then

$$s_{2k^m}(A) \le \min_{\psi_1,\ldots,\psi_{k^m-1}} \max_{\substack{\|f\|=1 \\ (f,\chi_0)=\ldots=(f,\chi_{k^m-1})=0}} \|Kf\| =$$

$$= \min_{\psi_1,\ldots,\psi_{k^m-1}} \max_{\substack{\|f\|=1 \\ (f,\chi_0)=\ldots=(f,\chi_{k^m-1})=0 \\ (f,\psi_1)=\ldots=(f,\psi_{k^m-1})=0}} \left\|(K-K_{k_1,\ldots,k_m})f\right\| \le$$

$$\le s_{k^m}(K-K_{k_1,\ldots,k_m}) \le k^{-\frac{m}{2}} \left\|K-K_{k_1,\ldots,k_m}\right\| =$$

$$= k^{-\frac{m}{2}} \left( \int_{\Omega_x}\int_{\Omega_y} \left|K(x,y)-K_{k_1,\ldots,k_m}(x,y)\right|^2 dxdy \right)^{\frac{1}{2}} =$$

$$= k^{-\frac{m}{2}} \left( \int_{\Omega_x} \left( \sum_{k_1=0}^{k-1}\cdots\sum_{k_m=0}^{k-1} \int_{\Omega_{k_1,\ldots,k_m}} \left|K(x,y)-K_{k_1,\ldots,k_m}(x,y)\right|^2 dy \right) dx \right)^{\frac{1}{2}} =$$

$$= k^{-\frac{m}{2}} \left( \sum_{k_1=0}^{k-1}\cdots\sum_{k_m=0}^{k-1} \int_{\Omega_{k_1,\ldots,k_m}} dy \left( \int_{\Omega_x} \left|K(x,y)-K_{k_1,\ldots,k_m}(x,y)\right|^2 dx \right) \right)^{\frac{1}{2}} = \qquad (2)$$

$$= k^{-\frac{m}{2}} \left( \sum_{k_1=0}^{k-1}\cdots\sum_{k_m=0}^{k-1} \int_{\Omega_{k_1,\ldots,k_m}} dy \left( \int_{\Omega_x} \left|K(x,y)-K_{k_1,\ldots,k_m}(x,y)\right|^r \times \right.\right.$$

$$\left.\left. \times \left|K(x,y)-K_{k_1,\ldots,k_m}(x,y)\right|^{2-r} \right) dx \right)^{\frac{1}{2}} \le$$

$$\le k^{-\frac{m}{2}} \left( \omega\left( K; \sqrt{\sum_{i=1}^{m}\left(\frac{d_i-c_i}{k}\right)^2} \right) \right)^{\frac{r}{2}} \left( \int_{\Omega_x} \left( \sum_{k_1=0}^{k-1}\cdots\sum_{k_m=0}^{k-1} \int_{\Omega_{k_1,\ldots,k_m}} \left|K(x,\sigma+ \right.\right.\right.$$

$$\left.\left.\left. +y_{k_1,\ldots,k_m}) - K_{k_1,\ldots,k_m}(x,y)\right|^{2-r} d\sigma \right) dx \right)^{\frac{1}{2}} =$$

$$= k^{-\frac{m}{2}} \left( \omega\left( K; \frac{d_y}{k} \right) \right)^{\frac{r}{2}} \left( \int_{\Omega_x} \left( \int_{\Omega_{k_1,\ldots,k_m}} \sum_{k_1=0}^{k-1}\cdots\sum_{k_m=0}^{k-1} \left|K(x,\sigma+ \right.\right.\right.$$

$$\left.\left.\left. +y_{k_1,\ldots,k_m}) - K_{k_1,\ldots,k_m}(x,y)\right|^{2-r} d\sigma \right) dx \right)^{\frac{1}{2}}.$$

Thus highlighted a factor that depends on the modulus of continuity. We estimate the last factor on the right side of (2) it transforms as follows:

$$\int_{\Omega_{k_1,\dots,k_m}} \sum_{k_1=0}^{k-1}\dots\sum_{k_m=0}^{k-1} \left|K(x,\sigma+y_{k_1,\dots,k_m})-K_{k_1,\dots,k_m}(x,y)\right|^{2-r} d\sigma =$$

$$=\prod_{j=1}^{m}\frac{b_j-a_j}{k}\sum_{k_1=0}^{k-1}\dots\sum_{k_m=0}^{k-1}\left|K(x,\sigma+y_{k_1,\dots,k_m})-K_{k_1,\dots,k_m}(x,y)\right|^{2-r} \le \qquad (3)$$

$$\le \frac{\prod_{j=1}^{m}(b_j-a_j)}{k^m}\sum_{k_1=0}^{k-1}\dots\sum_{k_m=0}^{k-1}\Big(\left|K(x,\sigma_{k_1,\dots,k_m}+y_{k_1,\dots,k_m})-K_{k_1,\dots,k_m}(x,y)\right|+$$

$$+\left|K(x,\sigma_{k_1,\dots,k_m}+y_{k_1,\dots,k_m})-K_{k_1+1,\dots,k_m+1}(x,y)\right|\Big)^{2-r} = O\left(\frac{1}{k^m}\ \underset{\Omega_x,\Omega_y}{V^{(2)}_{2-r}}(K;k)\right)$$

Substitution of (3) in (2) gives the desired result.

1. Similarly, a connection between the singular values of integral Hilbert–Schmidt operator with some kernel and the integral operator with Hilbert–Schmidt kernel, which is the partial derivative of the kernel. These results are obtained by applying the method of Krein, he proposed in the case of functions of one variable [8] (see also [5]). Consider the integral operator Fredholm kernel

$$\frac{\partial K(x,y_1,\dots,y_m)}{\partial y_1^{j_1}\dots\partial y_m^{j_m}} \in L^2(Q^m\times Q^n),\ j=\sum_{q=1}^{m} j_q .$$

**Theorem 2.**

$$s_N(K)=O\left(N^{-(j+1)}\left(\omega\left(K_{j_1,\dots,j_m},d_y N^{-1/m}\right)\right)^{r/2}\left(\underset{Q^{n,m}}{V_{2-r}}\left(K_{j_1,\dots,j_m};N^{\frac{1}{m}}\right)\right)^{1/2}\right) \qquad (4)$$

Let $K(x,y)$ and $M(x,y)$, $x\in\mathbb{R}^n$, $x\equiv(x_1,\dots,x_n)$, $y\in\mathbb{R}^m$, $y\equiv(y_1,\dots,y_m)$ – two kernels of integral operators $K$ and $M$ satisfying on $Q^d$ (1)

$$\int_{Q^m\times Q^n}\left|K(x,y)\right|^2 dxdy<+\infty;\ \int_{Q^m\times Q^n}\left|M(x,y)\right|^2 dxdy<+\infty \qquad (5)$$

Let note $M(x,y)=\frac{\partial}{\partial y_q}K(x,y),\ q=\overline{1,m}$, identifying values

$$K(x,y)\equiv K(x,y_1,\dots,y_m)\equiv K(x_1,\dots,x_n,y_1,\dots,y_m)$$

Let call $M(x,y)$ partial derivative in respect of $y_q$ in average of the kernel $K(x,y)$, if

$$\lim_{h\to 0}\int_{Q^n}\left|M(x,y)-\frac{K(x,y_1,...,y_{q-1},y_q+h,y_{q+1},...,y_m)}{h}-\frac{K(x,y_1,...,y_m)}{h}\right|^2 dx=0 \quad (6)$$

while $y_1,...,y_{q-1},y_{q+1},...,y_m$ are fixed. The kernels $K(x,y)$ and $M(x,y)$ by virtue of (5) are vector-functions of $X(y)$ and $Y(y)$, $y\in Q^m$, with results in $L^2(Q^n)$. The ratio (6) means that the vector function $Y(y)$ is a strong partial derivative $\frac{\partial X}{\partial y_q}$ of the vector-function $X(y)$ everywhere in $Q^m$. If $g(y)$, $y\in Q^m$, – numeric continuously differentiable function such that $g(y)=0$ for $y\in\partial Q^m$, denoting the projection of the cube $Q^m$ on the $(m-1)$–dimensional subspace that is perpendicular to the axis $Oy_q$, using $Q_q^{m-1}$, we obtain

$$\int_{Q^m} X(y)\frac{\partial g(y)}{\partial y_q}dy=\int_{Q_q^{m-1}} dy_1...dy_{q-1}dy_{q+1}...dy_n\int_{a_q}^{b_q} X(y)\frac{\partial g(y)}{\partial y_q}=$$

$$=\int_{Q_q^{m-1}} dy_1...dy_{q-1}dy_{q+1}...dy_n\left(X(y)g(y)\Big|_{a_q}^{b_q}-\int_{a_q}^{b_q}\frac{\partial g(y)}{\partial y_q}g(y)dy_q\right)=-\int_{Q^m}\frac{\partial X(y)}{\partial y_q}g(y)dy$$

where the integrals converge in norm $L^2(Q^m)$. Therefore, under these conditions with respect to $g(y)$ in the sense of mean convergence

$$\int_{Q^m} K(x,y)\frac{\partial g(y)}{\partial y_q}dy=-\int_{Q^m}\frac{\partial K(x,y)}{\partial y_q}g(y)dy \quad (7)$$

Operators defined by kernels $K$ and $M$, obviously, by (5) are Hilbert–Schmidt. Estimate the singular numbers of the operator $K$ using singular numbers of the operator $M$. To obtain this estimation we introduce the operator of integration over $q$ –th coordinate

$$J_q f=\int_t^b F(y_1,...,y_{q-1},y_q,y_{q+1},...,y_m)dy_q=\int_t^b f(y)dy_q\,. \quad (8)$$

$J_q f$ is a function of variables $t,y_1,...,y_{q-1},y_{q+1},...,y_m$ (that is a function of the variable $t$ with parameters $y_1,...,y_{q-1},y_{q+1},...,y_m$). We will omit the index $q$ until the operator $J_k$. Since the numbers $s_\mu^2(J)$ $(\mu=1,2,...)$ are the eigenvalues of the operator $J\times J$, then (omit parameters $y_1,...,y_{q-1},y_{q+1},...,y_m$) $\lambda=s_\mu^{-2}(J)$ are characteristic values of the integral equation: $\varphi(t)=\lambda\int_a^t\left(\int_v^b\varphi(u)du\right)dy$. It is clear that the integral equation is equivalent to a boundary value problem $\varphi''+\lambda\varphi=0$, $\varphi(a)=0, \varphi'(b)=0$. The general solution of equation has the form $\varphi(t)=C_1\cos\sqrt{\lambda}\,t+C_2\sin\sqrt{\lambda}\,t$, using initial conditions this gives a system of equations

$$\begin{cases} C_1\cos\sqrt{\lambda}\,a+C_2\sin\sqrt{\lambda}\,a=0, \\ -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,b+C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,b=0. \end{cases}$$

This system has a nontrivial solution if its determinant is zero, that is, $\lambda_\mu=\dfrac{\pi^2(2\mu-1)^2}{4(b-a)^2}$; $n=1,2,\dots$. implies that $s_\mu=\dfrac{2(b-a)}{\pi(2\mu-1)}$, $n=1,2,\dots$.

We denote $\mathcal{L}_0$ the subspace $L^2(a,b)$ of orthogonal to the function $e(t)\equiv 1$ through the $R$ – operator, projecting $L^2(a,b)$ on $\mathcal{L}_0$. A one-dimensional orthogonal projection $P=I-R$ is given by

$$(Pf)(t)=\frac{1}{b-a}\int_a^b f(t)dt,\ t\in[a,b]. \tag{9}$$

If $\mathrm{f}\in\mathcal{L}_0$, then $\mathrm{Pf}=0$. Consider the function $\mathrm{g}=\mathrm{Jf},\ \mathrm{f}\in\mathcal{L}_0$, i.e. $\mathrm{g(t)}=\int_t^b \mathrm{f(y)dy}$. We denote the subspace orthogonal to the virtue of (8) and equality $\mathrm{Pf}=0$ have $\mathrm{g(a)}=0$ besides $\mathrm{g(b)}=0$. Therefore, $g$ is a solution of boundary value problem $\left\{\dfrac{\mathrm{dg}}{\mathrm{dt}}=-\mathrm{f(t)},\mathrm{g(a)}=\mathrm{g(b)}=0\right\}$. Consider the operator $\mathrm{K_0}=\mathrm{KR}$. As $\mathrm{K}=\mathrm{KI}=\mathrm{K(R+P)}=\mathrm{KR}+\mathrm{KP}=\mathrm{K_0}+\mathrm{KP}$, then $\mathrm{K}-\mathrm{K_0}=\mathrm{KP}$ – a one-dimensional operator. Therefore, by Corollary 2.1 [5], Chapter II, § 2 $\mathrm{s}_{\mu+1}(\mathrm{K_0})\le\mathrm{s}_\mu(\mathrm{K})\le\mathrm{s}_{\mu-1}(\mathrm{K_0}),\quad \mu=2,3,\dots$.

Let $\mathrm{f}\in\mathrm{C(a,b)}$. Since $\mathrm{PRf}\equiv 0$ there is a continuously differentiable function $g(t)$ such that $\dfrac{\mathrm{dg}}{\mathrm{dt}}=-(\mathrm{Rf})(\mathrm{t}),\ \mathrm{g(a)}=\mathrm{g(b)}=0$. Then, by (7)

$$(\mathrm{K_0f})(\mathrm{t})=(\mathrm{KRf})(\mathrm{t})=-\mathrm{K}\left(\frac{\mathrm{dg}}{\mathrm{dt}}\right)=-\int_{\mathrm{Q^m}}\mathrm{K(x,y)}\frac{\partial\mathrm{g}}{\partial\mathrm{y_q}}\mathrm{dy}=$$

$$=\int_{Q^m}\frac{\partial K(x,y)}{\partial y_q}g(y_q)dy=(MJf)(t),$$

where $M$ – the Hilbert–Schmidt operator with kernel $M(x,y)=\dfrac{\partial K(x,y)}{\partial y_q}$.

Since $C(a,b)$ is dense in $L^2(a,b)$, then $K_0 = MJ$ due to Corollary 2.2 ([5], Chapter II, § 2) $s_{2\mu-1}(K_0) = s_{2\mu-1}(MJ) \le s_\mu(M)s_\mu(J) = \dfrac{2(b-a)}{\pi(2\mu-1)} s_\mu(M)$, and so $s_{2\mu}(K) \le s_{2\mu-1}(K_0) \le \dfrac{2(b-a)}{\pi(2\mu-1)} s_\mu(M)$. Let find for any number $N \in \mathbb{N}$ a number $l \in \mathbb{N}_0$ such that the inequalities $2^l \le N < 2^{l+1}$. Consider the integral Fredholm operator with the kernel

$$\frac{\partial K(x, y_1, ..., y_m)}{\partial y_1^{j_1} ... \partial y_m^{j_m}} \in L^2(Q^m \times Q^n),\ j = \sum_{q=1}^{m} j_q .$$

We have, using $j$ times last inequality, assuming at the same time $\mu = 2^{l-1}, ..., 2^{l-j+1}$ and letting natural operators with differentiated nucleus,

$$s_N(K) \le s_{2^l}(K) = O\left(\frac{1}{N} s_{2^{l-1}}(K_{1,0,...,0})\right) =$$

$$= O\left(\frac{1}{N^2} s_{2^{l-2}}(K_{2,0,...,0})\right) = ... = O\left(\frac{1}{N^k} s_{2^{l-j}}(K_{j_1,...,j_m})\right),$$

where $K_{j_1,...,j_m}(x,y) = \dfrac{\partial K(x, y_1, ..., y_m)}{\partial y_1^{j_1} ... \partial y_m^{j_m}}$.

Due to the fact that $2^{l-j} = 2^{l+1} \cdot 2^{-j-1} \ge \dfrac{N}{2^{j+1}} \ge \left[\dfrac{N}{2^{j+1}}\right]$ the inequality $s_N(K) = O\left(\dfrac{1}{N^j} s_{\left[N/2^{j+1}\right]}(K_{j_1,...,j_m})\right)$ is satisfied. Now it is easy to strengthen Theorem (1), extending it to the case when certain conditions are satisfied $j$-th derivative of the kernel of the integral operator.

**Theorem 3.**

$$s_N(K) = O\left(N^{-(j+1)}\left(\omega\left(K_{j_1,...,j_m}, d_y N^{-1/m}\right)\right)^{r/2} \times \left(\bigvee_{Q^{n,m}}{}^{2-r}\left(K_{j_1,...,j_m}; N^{1/m}\right)\right)^{1/2}\right).$$

In [5], Chapter III, § 10, we prove the following result concerning the integral operator from $L^2(a,b)$ to $L^2(a,b)$: if the kernel $\mathcal{A}(t,v)$ of the Hilbert–Schmidt $A$ in $L^2(a,b)$ has $l$–th derivative with respect to the second variable on average $\mathcal{A}_{0l}(t,v)$ which is a Hilbert–Schmidt operator, $\sum_{\mu=1}^{\infty}\mu^{2l}s_{\mu}^{2}(A)<+\infty$,

and therefore $s_{\mu}(A)=O\left(\mu^{-(l+1/2)}\right)$, $\mu\to+\infty$ and $A\in\mathfrak{S}_p$ at $p>\left(l+1/2\right)^{-1}$.

Below is the result transferred to the operator $K:L^2(Q^m)\to L^2(Q^n)$, and is considered a partial derivative, this derivative can be a kernel of the operator belonging to any ideal $\mathfrak{S}_r$ or ideal Matsaev $\mathfrak{S}_{\omega}$.

**Theorem 4.** *If the kernel* $K(x,y)$ *of the integral operator of Hilbert–Schmidt* $L^2(Q^m)\to L^2(Q^n)$ *is a derivative of order* $\gamma=\sum_{j=1}^{m}\gamma_j$ *, equal to* $\frac{\partial^{\gamma}K(x,y)}{\partial y_1^{\gamma_1}\ldots\partial y_m^{\gamma_m}}$ *, which is the kernel of the operator* $M\in\mathfrak{S}_r$ *, then* $K\in\mathfrak{S}_p$ at $p>\left(\gamma+r^{-1}\right)^{-1}$.

**Theorem 5.** *If the kernel* $K(x,y)$ *of the integral operator of Hilbert–Schmidt* $L^2(Q^m)\to L^2(Q^n)$ *is a derivative of order* $\gamma=\sum_{j=1}^{m}\gamma_j$ *, which is the kernel of the operator* $M\in\mathfrak{S}_{\omega}$*, then* $K\in\mathfrak{S}_p$ *at* $p>\gamma^{-1}$.

***Proof of Theorem 4.*** To prove the theorem we use the following fact (see [5], Chapter III, § 10 – a case $\beta=2$, [7] – the general case): if the sequence $\{s_{\mu}\}$ does not increase and $s_{\mu}>0$ when $\alpha\geq 0$, $\beta\geq 0$ from the fact that $\sum_{\mu}\mu^{\alpha}s_{\mu}^{\beta}<+\infty$ follows $s_{\mu}=O\left(\mu^{-(\alpha+1)/\beta}\right)$. Inequality for $s_N(K)$ and condition of the theorem: $\sum_{\mu}\mu^{jr}s_{\mu}^{r}(K)=O\left(\sum_{\mu}s_{\left[\mu/2^{j+1}\right]}^{r}(K_{j_1,\ldots,j_m})\right)=O(1)$. When $\alpha=jr$, $\beta=r$ we get that $s_{\mu}(K)=O\left(\mu^{-(jr+1)/r}\right)$. Hence $\sum_{\mu}s_{\mu}^{p}(K)=O\left(\sum_{\mu}\mu^{-p(j+1/r)}\right)<+\infty$. The theorem is proved.

***Proof of Theorem 5.*** If $M\in\mathfrak{S}_{\omega}$ (Matsaev ideal) then fixed $j$, $k\underset{\cap}{\overset{\cup}{}}\mu$ we have $\sum_{\mu}\mu^{j-1}s_{\mu}(K)=O\left(\sum_{\mu}\frac{2^{j+1}}{\mu}s_{\left[\mu/2^{j+1}\right]}^{r}(K_{j_1,\ldots,j_m})\right)=O(1)$. Applying Proposition, used in the proof of Theorem 4, at $\alpha=j-1$, $\beta=1$ we have

$s_\mu(K) = O\left(\mu^{-j}\right)$. Hence $\sum_\mu s_\mu^p(K) = O\left(\sum_\mu \mu^{-pj}\right) < +\infty$. The theorem is proved.

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