THE APPROXIMATE ANALYTICAL METHOD FOR CALCULATION OF NONLINEAR DEFORMATION OF SHELLS BASED ON 2-D PADÉ APPROXIMANTS

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Approximate analytic integration of nonlinear differential equations of solid mechanics and, in particular, the theory of flexible elastic shells in most practical cases is based on the method of continuation of solution on the artificial parameter [4]. Recently, the most frequently used two kinds of it: the Adomian's decomposition method (ADM) [1, 6] and homotopy analysis method (HAM) [8, 9]. Both methods have a significant impact on development of the theory of nonlinear ODEs analytical solution.

It is worth mentioned that ADM and HAM can be satisfactorily applied only with an effective method of summation. The most natural analytical continuation method is Padé approximants (PAs) [2, 4, 5]. Recently, the method of PAs for single-variable functions (1-D PAs) has been successfully extended to the approximation of two variable functions (2-D PAs) [5, 11]. In the case of ODEs both methods can be combined on the basis of a new approach, the development of which this work is devoted to.

1. Modified method of the parameter continuation. The modified method of the parameter continuation (MMPC) consists of perturbation technique of special form and the analytical continuation of obtained approximations by PAs. In vicinity of regular point in the interval $\Omega:\xi\in]0,1[$ any ODE or system of ODEs may be represented by a normal system of ODEs of the first order in respect to the unknown functions $u_i=u_i(\xi), i=\overline{1,n}$ with the BC on the bounds $\partial\Omega$:

$$Lu_i+R_i(\xi,u_1,...,u_n)+N_i(\xi,u_1,...,u_n)=g_i(\xi),$$
$$G_i(u_1,...,u_n)\big|_{\partial\Omega}=0,\ L=\frac{d}{d\xi}, i=\overline{1,n}, \tag{1}$$

Here *L*, R_i are the linear and N_i, G_j are the non-linear differential operators. We assume also that point $\xi_0=0$ belongs to closure Ω, and R_i, N_i and G_j are the holomorphic functions for $\{u_i\}_{i=1}^{n}$.

MMPC coincides with the HAM for the case, when N_1 doesn't contain Lu_1, and with the ADM – when $g\equiv 0$, and thus generalizes them. The method does not imply the introduction of «trial» functions that satisfy the BC, they will be satisfied in successive approximations, and this gives us an opportunity to solve the BVP with complicated BCs [2]. To implement the MMPC, we introduce parameter ε as follows

$$Lu_i = \varepsilon\left(g_i - R_i\left(u_1,...,u_n\right) - N_i\left(u_1,...,u_n\right)\right), u_i = \sum_{j=0}^{\infty} u_{ij}^{M} \varepsilon^j,$$
$$G_i(u_1\big|_{\partial\Omega},...,u_n\big|_{\partial\Omega})\big|_{\partial\Omega} = 0,\ i = \overline{1,n} \tag{2}$$

Substituting power series into equations and splitting it with respect to the powers of ε, after summation of the coefficients with the same degrees of ξ for $\varepsilon = 1$ we get

$$\begin{aligned}
u_i = {} & \xi^0\left(u_i\big|_{\partial\Omega} + ...\right) + \xi^1\left(g_{i0} - \sum_{r=1}^{n}\left(N_{ir}^0 u_r\big|_{\partial\Omega} + \frac{1}{2!}\sum_{p=1}^{n} N_{irp}^0\, u_r\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega} + ...\right) + ...\right) + \\
& + \xi^2\left(\frac{g_{i1}}{2} - -\frac{1}{2}\sum_{r=1}^{n}\left(N_{ir}^1\, u_r\big|_{\partial\Omega} + \frac{1}{2!}\sum_{p=1}^{n} N_{irp}^1\, u_r\big|_{\partial\Omega}\, u_p\big|_{\partial\Omega} + ...\right) - \right. \\
& - \sum_{r=1}^{n}\left(N_{ir}^0\left(\frac{g_{r0}}{2} - \left(\frac{1}{2}\sum_{l=1}^{n} N_{rl}^0\, u_l\big|_{\partial\Omega} + \frac{1}{2!}\sum_{q=1}^{n} N_{rlq}^0\, u_l\big|_{\partial\Omega}\, u_q\big|_{\partial\Omega} + ...\right) + \right.\right. \\
& + \frac{1}{2!}\sum_{p=1}^{n} N_{irp}^0\left(u_p\big|_{\partial\Omega}\left(\frac{g_{r0}}{2} - \frac{1}{2}\sum_{l=1}^{n}\left(N_{rl}^0\, u_l\big|_{\partial\Omega} + \frac{1}{2!}\sum_{q=1}^{n} N_{rlq}^0\, u_l\big|_{\partial\Omega}\, u_q\big|_{\partial\Omega}\right)\right) + \right. \\
& \left.\left.\left. + u_r\big|_{\partial\Omega}\left(\frac{g_{p0}}{2} - \frac{1}{2}\sum_{l=1}^{n}\left(N_{pl}^0\, u_l\big|_{\partial\Omega} + \frac{1}{2!}\sum_{q=1}^{n} N_{plq}^0\, u_l\big|_{\partial\Omega}\, u_q\big|_{\partial\Omega}\right)\right)\right) + ...\right) + ...\right) + ...,\quad i = \overline{1,n}.
\end{aligned} \tag{3}$$

Analysis of the obtained approximation suggests that, in contrast to the ADM and HAM, it gives the exact value of the coefficients in the power of the independent variable to the extent equal to the order of approximation (taking into account the expansion in power series of expressions in the equation). This guarantees stability of the computation with a limit-order approximation of the independent variable. MMPC approximation is simpler than ADM and HAM. The approximation thus obtained is converted to 2-D PA.

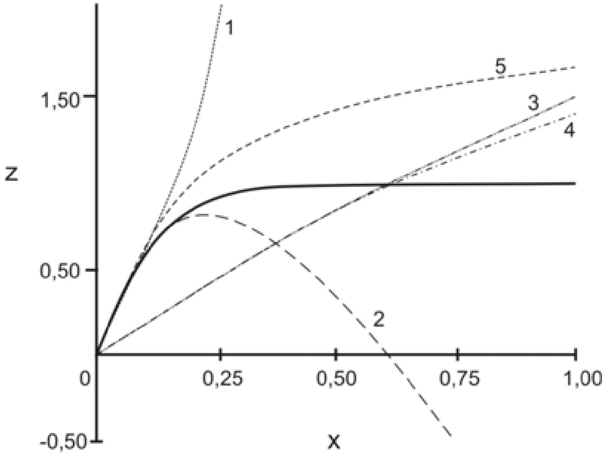
The proposed approach can be used to the nonlinear problems of plates and shells theory. The equations of static of geometrically nonlinear thin-walled structures can be reduced to the resolving equations, which contain the products and squares of the desired functions and their derivatives [10].

2. Using the PAS. If the equations are solved in respect to the highest derivative, the coefficients of ADM and HAM approximants with the same degree of variable solutions ADM and HAM converge to each other as far as the order of approximation increases. It was shown in [2] that the solution of the ADM converges to the decomposition of exact solution in the Taylor series in the area of its holomorphy in the vicinity of $x = 0$. That is the reason that the same properties will have a solution of HAM in the case when the equation is in a normal form. This allows to use meromorphic continuation in the form of PAs [5]. For the ADM such a continuation procedure was proposed in [2]. Later, this approach was developed by a number of authors [6], and was named modified Adomian's decomposition method and PAs (MADM-Padé). Thus, it is possible to use PAs to HAM with modifications, by decomposition of nonlinear terms in the series as for the independent variable, so for the desired function (MHAM-Padé).

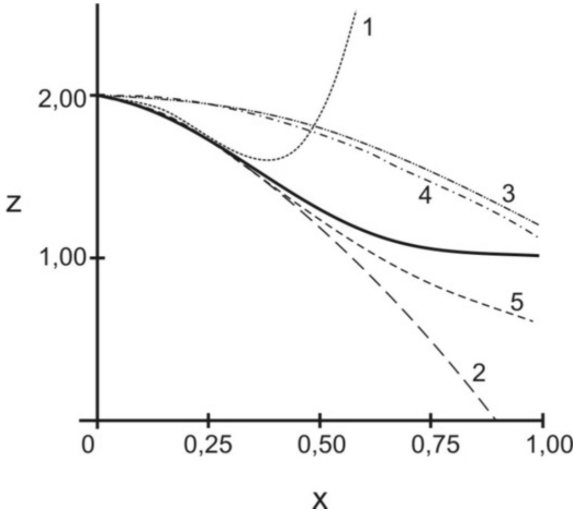
2-D PAs in the form proposed by V. Vavilov [11] is very promising for the use as an analytical continuation. This technique allows us to choose the coefficients of 2-D Taylor series for construction of an unambiguous 2-D PA with a given structure of the numerator and denominator, as well as ensures optimal PAs features in the sense of the Theorem of Montessus de Ballore–type. This means homogenous convergence of PA to the approximated function with an increase of the degree of the numerator and the denominator in all points of its meromorphy area. It should be noted that direct application of 2-D PAs does not lead to the anticipated merging of 1-D approximations. This is due to the initial requirements to the 2-D approximation to ensure its transition to 1-D in the case when the second variable is equal to zero [11]. At the same time as for the method of parameter continuation it is necessary to ensure such a transition when the parameter is equal to one.

3. Examples of numerical results. We consider three types of PAs – on the independent variable $z^{(x)}$, on the specified parameters $z^{(\varepsilon_1)}$, and 2-D. Typical behavior of the approximations for the BVP, where natural small parameter ε is the factor at the highest derivative, is shown in Fig. 1a for $\varepsilon = 0.1$. The ADM approximation well describes the exact solution only for a distance which is comparable with the value of the natural small parameter ε. Despite the fact that the error of solutions of HAM is substantially lesser than the ADM, HAM does not accurately reflect the nature of solutions, namely the phenomenon of boundary layer in the vicinity of zero. At the same time, PAs for the ADM approximations for independent variable and PAs for the MMPC (1-D and 2-D) give satisfactory qualitative and quantitative results. Similar results give us an analysis of approximations whose coefficients are given depending on the variable for $\varepsilon = 0.2$ (Fig. 1b). The graphs show that the solution is well described by the HAM approximation and MHAM–Padé «in average», and badly – in the boundary layer.

The ADM approximation and MADM–Padé, on the contrary, is in good agreement with the behavior of solution in the vicinity of zero and in the bad one – on the stationary part. At the same time, 1-D and 2-D PAs, based on approximations of the MMPC, well described the solution in the whole interval.



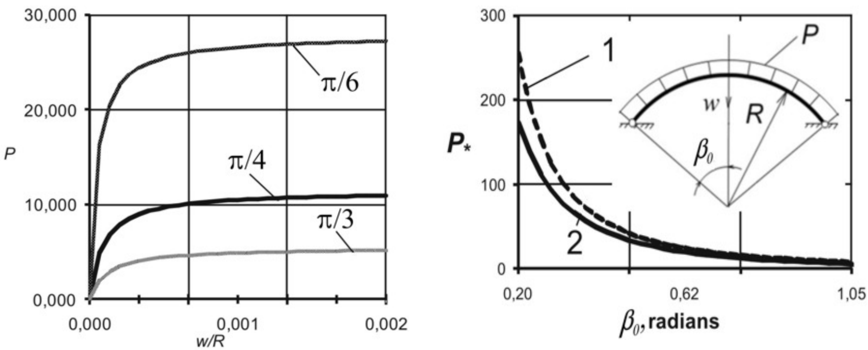
a – $\varepsilon z' + z = 1;\ z(0) = 0;\ x \ge 0$



b – $\varepsilon z' + xz = x;\ z(0) = 2;\ x \ge 0$

Fig. 1. The exact solution (solid line) and approximate solutions (1 – three terms $z^{(x)}$ for ADM, 2 – $z^{(\varepsilon_1)}$ for ADM, 3 – three terms $z^{(x)}$ for HAM, 4 – $z^{(x)}$ for HAM, 5 – 2–D Padé for MMPC, ADM, HAM)

4. Calculation of nonlinear deformation of shells. The MMPC was applied to calculate the deformation and stability of a long flexible elastic circular cylindrical shell of radius R with half the central angle β_0 in the case of cylindrical bending under uniform external pressure with simple support of the longitudinal edges. The system of resolving equations in normal form is given in [7]. Dependencies "intensity of pressure P – deflection w/R " for the top cross-section of shell at different angles and dimensionless flexibility $C = 10^{-4}$ and "intensity of the limit load P_L – size of half angle β_0" is shown in Fig. 2. For comparison, Fig. 2b also shows the dependence of the critical loads for inextensible shell obtained Timoshenko [7]. We see that dependences are in good agreement, while consideration of deformation of the longitudinal axis substantially affects the value of critical loads of construction.



a b

Fig. 2. Calculation of shell under uniformly distributed radial load. a) the dependence of the intensity of pressure P from the deflection w/R for different values of β_0 (the value of β_0 is indicated by the curves), b) the dependence of limit loads P_b from β_0 (1 – data [7], 2 – calculation).

The proposed method can be used in combination with the known asymptotic method. Consider the free vibrations of a flexible elastic circular cylindrical shell of radius R, thickness h and length L, backed by a set of uniformly stringers, with simple support at the ends. The calculation is based on mixed dynamical equations of the theory of shells after splitting them in powers of natural small parameters [3, 4]. Governing equations can be reduced by the

Bubnov – Galerkin method to the Cauchy problem with respect to $\xi = f_1 / R$ on $t_1 = t\sqrt{B_1/\rho R^2}$ (all the symbols are taken in accordance with [3])

$$\ddot{\xi} + \alpha\xi\left[\left(\dot{\xi}\right)^2 + \xi\ddot{\xi}\right] + A_1\xi + A_2\xi^3 + A_3\xi^5 = 0,\ t_1 = 0 : \xi = f, \dot{\xi} = 0 \quad (4)$$

Application of the method to the problem (4) gives the approximation of second order for the artificial parameter for the frequency Ω of nonlinear oscillations. It is seen that the oscillations are no isochronous. This agrees well with previous results (Fig. 3), while is significantly reduced the volume of computations (in [3] to obtain similar results the approximation of fourth order is taken).

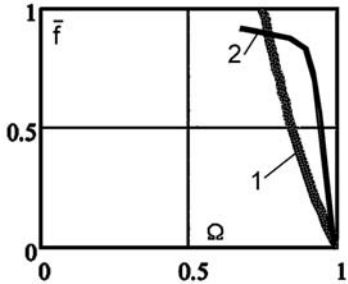


Fig. 3. The dependence of the oscillation frequency of stringer shell from the amplitude of the initial disturbance. (1 – according to the proposed method, 2 – data from [3])

5. Conclusions. Proposed method of the parameter continuation (MMPC) enables simplification of the calculations both at the stage of constructing the model, and also within its continuation due to the precise values of the Taylor coefficients for the solution of the degree which is not exceeding the number of approximation. It was concluded that the application of 1-D and 2-D PAs is justified if it is applied to polynomials, which depend on the variable of integration. It has been shown that 2-D PAs for the independent variable and for the artificial parameter used the scheme of V. Vavilov provides a satisfactory quality for the approximation behavior and minimizes its error. A study of numerical results for model examples confirms the advantage of MMPC approximations and shows that the application of PAs provides them with sufficient accuracy in the studied area. Calculations of nonlinear deformation and stability of elastic flexible circular cylindrical shell under uniform external pressures and of free oscillations of simply supported stringer shell demonstrate the efficiency and accuracy of the proposed method.

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