ANALYTIC EXTENSION OF RIEMANNIAN MANIFOLDS AND GENERALISATION OF COMPLETENESS

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Analytic extension of Riemannian analytic manifold M is another Riemannian analytic manifold N with imbedding $i: M \to N$ such that $i(M) \subset N$ but $i(M) \neq N$. So we can consider M as an open submanifold of N. It is natural to be interested in maximal extension i.e. extended to so called nonextensible manifold N which can not be extended furthermore. Definition of nonextensible manifold is given in monograph of Kobayashi and Nomizu [1]. Riemannian manifold M is called nonextensible if it cannot be isometrically imbedded in another Riemannian manifold ***N*** as a proper submanifold. It is proved in [1] that every complete manifold is nonextensible.

But every Riemannian analytic manifold has a lot of unnatural nonextensible analytic extensions. For example simply connected covering of right half plain without set of points $(1/n; k/n)$, $n \in N, k \in Z$, is maximal extension of a circle with Euclidian metric.

More appropriate analytical extension was made in thesis of G.H. Smith [2]. For metric without Killing vector fields there was built analytic extension of some small ball to simply connected oriented Riemannian manifold M uniquely defined by following properties:

a) M is nonextensible;

b) there is no preserving orientation isometry $\varphi: U \to M$ with fixed point x_0, $\varphi(x_0) = x_0$, $U \subset M$ — open subset;

c) if for some discrete group of isometries Γ there exists analytic extension N of factor manifold M/Γ than property b) is not valid for N.

The manifolds without Killing vector fields where investigated by author [3]. There was constructed analytic extension M of small Riemannian ball without Killing vector fields with the following properties:

a) M is nonextensible:

b) there is no exist nonidentical isometry $\varphi: U \to V$ between open subsets of M.

Manifold with properties a) and b) is called absolutely nonhomogeneous and for any metric without Killing vector fields there exists unique extension to such manifold.

It is difficult to find some properties in order to define the most appropriate unique maximal analytic extension of Riemannian manifold. with Killing vector fields. But something similar to the above mentioned properties can be found in case when algebra of all Killing vector fields has no center. This case was investigated in the work of author [4].

So let's consider a ball with Riemannian analytic metric whose algebra Lie of all Killing vector fields has no center. Such metrics include not only absolutely nonhomogeneous manifolds but also the manifolds with large symmetry. So symmetric spaces have algebra lie of all Killing vector fields without center. In that case we can offer some generalization of completeness.

Definition 1. Riemannian analytic oriented manifold M is called *quasicomplete* if satisfy the following properties:

a) M is nonextensible:

b) there is no nonidentical preserving orientation isometry $\varphi : U \to V$ between open subsets of M .with fixed point x_0, $\varphi(x_0) = x_0$.

Theorem 1. *Every small Riemannian analytic ball whose algebra Lie of all Killing vector fields has no center there exist a unique extension to quasicomplete manifold.*

Theorem 2. *Any isometry $\varphi : U \to V$ between two open subsets $U \subset M$ and $V \subset N$ of quasicomplete manifolds M and N can be extended to isometry $\varphi : M \to N$.*

Theorem 2 can be applied to homogeneous spaces. Let ζ algebra Lie of all Killing vector fields on a locally homogeneous Riemannian analytic ball U and $\eta \subset \zeta$ is a stationary subalgebra so that $\operatorname{codim} \zeta/\eta = \dim M$. G is a simply connected group corresponding algebra ζ and $H \subset G$ is a subgroup corresponding to subalgebra $\eta \subset \zeta$. Than by theorem 2 quasicomplete extension of U is homogeneous space G/H. This result can be formulated and proved in algebraic terms.

Theorem 3. *Let ζ algebra Lie of all Killing vector fields on Riemannian analytic manifold M with fixed point $x_0 \in M$, $\eta \subset \zeta$ is a stationary subalgebra and G — simply connected group corresponding to algebra ζ. Assume also* *that* $\dim M = \dim\zeta - \dim\eta$. *Than subgroup* $H \subset G$ *corresponding subalgebra* $\eta \subset \zeta$ *is closed in* G.

Proof. Let's suppose that H is not closed and $\overline{H}$ is closure of H in G and $\overline{\eta} \subset \zeta$ — algebra of group $H \subset G$. Consider vector field $Z \in \overline{\eta}$, $Z \notin \eta$ and one parameter subgroup $\overline{h_t}$ generated by vector field Z. As we can identify some ball in M with some ball in ζ/η the inner automorfism hgh^{-1} generates local isometry h of M with fixed point x_0. Local transformation of M generated by inner automorfism $\overline{h}_t^{-1} g\overline{h}_t$ is local isometry as it is limit of isomeries generated by inner automorfisms $h_n g h_n^{-1}$. So right multiplication gh_t generates local isometry too. This local isometry commute with every local isometry generated by left multiplication. So family of local isometries generated by right multiplication by h_t commute with group G. Consequently vector field Y tangent to that family belongs to the center of ζ. This contradicts to the conditions of theorem.

Definition of quasicomplete manifold has two shortages. It is not valid for all metrics. There are balls of complete Riemannian analytic manifolds whose quasicomplete extensions are not complete. For example let's consider metric $ds^2 = f\left(z;\overline{z}\right)dzd\overline{z}$ on complex where $f\left(-z;-\overline{z}\right) = f\left(z;\overline{z}\right)$, ds^2 has no other symmetries and defines completes two dimensional manifold. But quasicomplete manifold of this metric is factor manifold. by group of two elements $\{1;-1\}$ with coordinate $w = z^2$ .and metric $ds^2 = \left(f(\sqrt{w};\sqrt{\overline{w}})/4|w|\right)dwd\overline{w}$. This defines Riemannian manifold on a plain without point $(0;0)$ and so this manifold is not complete.

Let's give another generalization of completeness.

Definition 2. Simply connected Riemannian analytic manifold is called *pseudocomplete* if there is no locally isometric covering $f: M \to N$ where N is simply connected Riemannian analytic manifold of the same dimension and $f(M) \neq N$.

It can be easily proved that if ball with Riemannian analytic analytically extended to complete manifold than pseudocomplete extension of this ball is unique and complete too. In favor of definition of pseudocomplete manifold should be noticed that pseudocomplete manifold exist for any locally given analytic Riemannian metric. But in general pseudocomplete manifold is not unique for many metrics. Let's give two examples.

Consider Lie group G and no closed subgroup $H \subset G$. Then locally we can define manifold G/H and introduce Riemannian analytic metric such that algebra Lie of Killing vector fields contains algebra Lie ζ of group G and algebra Lie of stationary Killing vector fields contains algebra Lie η of group H. Then there are a lot of pseudocomplete manifolds of this metric. As another example Riemannian analytic metric $ds^2 = \frac{f(z;z)}{\sqrt{1+|z|^6}} dz d\bar{z}$ on a complex sphere with singularity at point $z = \infty$. This manifold is pseudocomplete. In order to obtain another pseudocomplete manifold of the same locally given metric remove singularity at point ∞ by covering sphere with coordinate z by sphere with coordinate w, $z = w^2 + a$, $a \in C$.

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