ON UNITARY DIVISORS $\tau_k^*(n)$ OF INTEGER NUMBERS

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We investigate the distribution of value of the function $\tau_k^*(n)$ that defines the number of ways to write *n* as the product of *k* relatively primes. The nontrivial estimations were obtained for summatory function $\tau_k^*(n)$ for different values of *k*.

Let $\tau(n)$ is the number divisors of positive integer *n*, $\omega(n)$ is the number of different prime divisors of *n* .

J.M. de Koninck and A.Ivic [1] , on applying analytical methods , obtained the asymptotic formula for the summatory function

$$G(x) := \sum_{n\le x} \tau(n)\omega(n) = 2x\log x \log\log x + Ax\log x + O(x), \qquad (1)$$

where $A = 2(\sum_p (\log(1-\frac{1}{p}) + (\frac{1}{p} + \frac{3}{2p^2} + \frac{4}{2p^3} + ...)(1-\frac{1}{p})^2) + \Gamma'(2)$

where Γ is gamma function of Euler .

In 2003 J. M. de Koninck and I.Katai [2] amplified this asymptotic form

$$\sum_{n\le x} \tau(n)\omega(n) = 2x\log x\log\log x + c_1 x\log x + c_2 x\log\log x + c_3 x + O(\frac{x}{\log x}),$$

where $c_1 = 2\sum_p (\log(1-\frac{1}{p}) + \frac{1}{p}) - \sum_p \frac{1}{p^2} + 2(\gamma - 1)$

$$c_2 = 4\gamma - 2$$

$$c_3 = 4\gamma^2 + (4\gamma - 2)\sum_p (\log(1-\frac{1}{p}) + \frac{1}{p}) + (1-2\gamma)\sum_p \frac{1}{p^2} +$$

$$+2\sum_p \frac{8p^4 - 11p^3 + 18p^2 - 17p + 6}{p^3(p-1)^3}\log p$$

The aim of the present work is investigation of distribution of the average values of the functions of unitary divisors $\tau_k^*(n)$ weighted by the function $\omega(n)$, where $\tau_k^*(n) = \sum_{\substack{n=n_1\cdot\ldots\cdot n_k \\ (n_i,n_j)=1 \\ i\neq j}} 1$.

We use following standard denotions: *p* is for prime numbers , $\mu(n)$ is Möbius function, $\phi(n)$ is Euler function, (a,b) is the greatest common divisor of positive integer of a and b, $s = \sigma + it$ is the complex variable $\sigma = \mathrm{Re}\, s, t = \mathrm{Im}\, s$.

$\operatorname{res}_{s=a} F(s)$ is the residual of the function $F(s)$ in the point $s = a$, the Vinogradov symbol << is equivalent to Landau symbol O, γ is Euler constant, $\exp U := e^u$.

Lemma1. The *Riemann zeta function* $\zeta(s)$ *for* $s = \sigma + it$ *satisfies the following estimations*

$$\zeta(s) << \begin{cases} 1, \text{if } \sigma \geq 1+\varepsilon \\ \log|t|, \text{if } 1 \leq \sigma \leq 1+\varepsilon \\ |t|^{(1-\sigma)/3}\log|t|, \text{if } 1-\varepsilon \leq \sigma \leq 1 \\ |t|^{(1-\sigma)/3}, \text{if } \frac{1}{2} - \frac{c}{(\log|t|)^{2/3}(\log\log|t|)} \leq \sigma \leq 1-\varepsilon \\ |t|^{(3-4\sigma)/3}, \text{if } 0 < \sigma \leq \frac{1}{2} \end{cases} \qquad (2)$$

where $c > 0$ *is some absolute constant* , $\varepsilon > 0$ *is arbitrary small number. The constant in symbol* $<<$ *can depend on* ε *only* (see , A. Ivic [3]).

Lemma2. *For* $t \to \infty$ *and* $\varepsilon > 0$ *we have*

$$\int_1^T \left|\zeta(\frac{1}{2}+it)\right|^2 dt = T\log\frac{T}{2\pi} + (2\gamma-1)T + O(T^{\frac{1}{2}+\varepsilon}) \qquad (3)$$

$$\int_1^T \left|\zeta(\frac{1}{2}+it)\right|^4 dt = TP_4(\log T) + O(T^{\frac{7}{8}+\varepsilon}) \qquad (4)$$

$$\int_1^T \left|\zeta(\frac{1}{2}+it)\right|^k dt << T^{m(k)+\varepsilon} \qquad (5)$$

where

$$m(k) = \begin{cases} 1+\frac{k-4}{8}, \text{if } 4 \leq k \leq 12 \\ 2+\frac{3}{22}(k-12), \text{if } 12 \leq k \leq 178/3 \\ 1+\frac{35}{216}(k-6), \text{if } k \geq 178/3 \end{cases}$$

with constant implied in symbol $<<$ *depending on* ε *only* .(see [3])

Lemma3. *Let* $k \geq 2$ *is the fixed positive integer . Then in the domain* $\operatorname{Re} s > 1$ *, we have*

$$F_k(s) := \sum_{n=1}^{\infty} \frac{\tau_k^*(n)}{n^s} = \frac{\zeta^k(s)}{(\zeta(2s))^{k(k-1)/2}} G_k(s) \qquad (6)$$

where $G_k(s)$ is regular in the domain $\mathrm{Re}\, s \geq \frac{1}{3}+\varepsilon$ and in the given domain it is defined by the absolutely convergent Dirichlet series

$$G_k(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}, |g(n)| << n^{\varepsilon} .$$

Proof. Cause the function $\tau_k^*(n)$ is multiplicative and $\tau_k^*(p^n)=k$ we obtain $\tau_k^*(m)=k^{\omega(n)}$ **.** Besides , for $\mathrm{Re}\, s>1$ we have

$$1+\frac{k}{p^s}+\frac{k}{p^{2s}}+...=\frac{1}{(1-\frac{1}{p^s})^k}(1+\frac{k}{p^s}+\frac{k}{p^{2s}}+...)(1-\frac{k}{p^s}+\frac{k(k-1)}{p^{2s}}+...)=$$

$$=\frac{1}{(1-\frac{1}{p^s})^k}(1-\frac{k(k-1)}{2p^{2s}}+O(\frac{1}{p^{3s}}))=(1-\frac{1}{p^s})^{-k}(1-\frac{1}{p^{2s}})^{k(k-1)/2}(1+O(\frac{1}{|p^{3s}|}))$$

with the constant in symbol O depending only on *k* .

The assertion of Lemma 3 is proved .

In the special case, for *k*=2 we get $\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\frac{\zeta^2(s)}{\zeta(2s)}$. Indeed,

$$\frac{\zeta^2(s)}{\zeta(2s)}=\prod_p\frac{1-1/p}{(1-1/p^s)^2}=\prod_p\frac{1+\frac{1}{p^s}}{1-\frac{1}{p^s}}=\prod_p(1+\frac{1}{p^s})(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+...)=\prod_p(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+...).$$

From the other side $\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n^s}=\prod_p(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+...)$, whence we obtain the proof of the special case.

Let $D(x)=\sum_{n\leq x}\tau_2(n)=x\log x+(2\gamma-1)x+\Delta(x)$,

$$D^*(x)=\sum_{n\leq x}\tau_2^*(n)=\frac{1}{\zeta(2)}(x\log x+(2\gamma-1)x-2x\frac{\zeta'(2)}{\zeta(2)})+\Delta^*(x) .$$

The best estimation for $\Delta(x)$ gave M.Huxley[4]

$$\Delta(x)=O(x^{\theta}(\log x)^{\theta_1}) \tag{7}$$

where $\theta=\frac{131}{416}, \theta_1=\frac{315}{146}$ and the best estimation for $\Delta^*(x)$ D.Suryanarayan and V.Siva Rama Prasad obtained [5] .

$$\Delta^*(x)=O(x^{1/2}\exp(-A(\log x)^{3/5}(\log\log x)^{1/5}), A>0-\mathrm{const} . \tag{8}$$

We note that estimation (8) essentially depends on the bound of nontrivial zeros of Riemann zeta function and for the current moment can not be improved. However, as was shown in [5] , supposing that Riemann hypothesis is hold, estimation (8) for $\Delta^*(x)$ has the form

$$\Delta^*(x) = O(x^{(2-\theta)/(5-4\theta)} \exp(B\log x(\log\log x)^{-1}), B > 0 - \text{const} \qquad (9)$$

where θ is taken from (7) .

Theorem 1. *On supposing that Riemann hypothesis is hold , for any positive ε we obtain*

$\Delta^*(x) = O(x^{(1/(3-2\theta))+\varepsilon})$ (*the constant in symbol O depends only on* ε)

Proof. For $\operatorname{Re} s > 1$ we have $F(s) = \sum_{n=1}^{\infty} \frac{\tau_2^*(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n-1}^{\infty} \frac{\tau_2(n)}{n^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}$

and that is why

$$\tau_2^*(n) = \sum_{md^2=n} \tau_2(m)\mu(d) \qquad (10)$$

Thus, $D^*(x) = \sum_{n\le x} \tau_2^*(n) = \sum_{md^2 \le x} \tau_2(m)\mu(d)$. We take $y, 1 \le y \le x^{1/2}$ (more precisely it is to be chosen later). Then,

$$D^*(x) = \sum_{d\le y} \mu(d) \sum_{m \le x/d^2} \tau_2(m) + \sum_{y<d\le x^{1/2}} \mu(d) \sum_{m\le x/d^2} \tau_2(m) = S_1 + S_2 . \qquad (11)$$

We can write

$$S_1 = \sum_{d\le y} \mu(d) D(\frac{x}{d^2}) = \sum_{d\le y} \mu(d)[(\frac{x}{d^2}(\log x - 2\log d + 2\gamma - 1) + O((\frac{x}{d^2})^{\theta})] =$$

$$x\log d \sum_{d\le y} \frac{\mu(d)}{d^2} - 2x \sum_{d\le y} \frac{\mu(d)\log d}{d^2} + (2\gamma-1)x \sum_{d\le y} \frac{\mu(d)}{d^2} + O(x^{\theta} \sum_{d\le y} \frac{1}{d^{2\theta}}).$$

We take into account that on supposing that Riemann hypothesis in domain $\operatorname{Re} s > \frac{1}{2} + \varepsilon, |s-1| < 1$ is hold , then $\zeta(s) << |t|^{\varepsilon} + 1, \zeta^{-1}(s) << |t|^{\varepsilon} + 1$.

$$\sum_{d\le y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O(y^{(1/2)-\operatorname{Re} s+\varepsilon}(|t|^{\varepsilon} + 1)) \qquad (12)$$

(Proof see in [5]). Consequently , from (11) we get immediately

$$S_1 = (x\log x + (2\gamma-1)x)(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d>y} \frac{\mu(d)}{d^2}) - 2x(\sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d^2} - \\ - \sum_{d>y} \frac{\mu(d)\log d}{d^2}) + O(x^{\theta}(y^{1-2\theta} + 1)) \qquad (13)$$

But from (12) we have

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)}, \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \log d = \frac{\zeta'(2)}{\zeta^2(2)}, \sum_{d>y} \frac{\mu(d)}{d^2} << y^{-\frac{3}{2}+\varepsilon}, \\ \sum_{d>y} \frac{\mu(d)}{d^2} \log d << y^{-\frac{3}{2}+2\varepsilon} \qquad (14)$$

Thus,

$$S_1=(x\log x+(2\gamma-1))\frac{1}{\zeta(2)}-2x\frac{\zeta'(2)}{\zeta^2(2)}+O(x^{\theta}(y^{1-2\theta}))+O(xy^{-\frac{3}{2}+2\varepsilon}). \quad (15)$$

For $\operatorname{Re}s>1$ we denote $f_y(s)=\zeta^{-1}(s)-\sum_{d\le y}\frac{\mu(d)}{d^s}$. It is obvious that $f_y(s)=\sum_{d<y}\frac{\mu(d)}{d^s}$. But for $\operatorname{Re}s>1$ we obtain

$$f_y(2s)\zeta^2(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}, a_n=\sum_{\substack{md^2=n\\ d>y}}\tau(m)\mu(d)\,. \quad (16)$$

That is why $S_2=\sum_{n\le x}a_n, a_n \ll n^{\varepsilon}$. On applying Perron formula for $c>1, T>1$ we obtain

$$S_2=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}f_y(2s)\zeta^2(s)\frac{x^s}{s}ds+O(\frac{x^c}{T(c-1)^2}) \quad (17)$$

We remove the contour of integration on the straight line $\operatorname{Re}s=\frac{1}{2}+\varepsilon, |s-1|<\varepsilon$.

Thus we get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}f_y(2s)\zeta^{2(s)}\frac{x^s}{s}ds=\operatorname*{res}_{s=1}(f_y(s)\zeta^2(s)\frac{x^s}{s})+$$
$$+\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon+iT}^{c+iT}f_y(2s)\zeta^{2(s)}\frac{x^s}{s}ds-\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{c-iT}f_y(2s)\zeta^{2(s)}\frac{x^s}{s}ds+ \quad (18)$$
$$+\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}f_y(2s)\zeta^{2(s)}\frac{x^s}{s}ds=(x\log x-x)f_y(2)+2x\gamma f_y(2)+2xf_y{}'(2)+$$
$$+O(y^{-\frac{1}{2}}x^{\frac{1}{2}+8\varepsilon})$$

Besides,

$$f_y(2s)\ll y^{-\frac{1}{2}}(|t|^{\varepsilon}+1), \qquad f_y(2s)\zeta^2(s)\ll y^{-\frac{1}{2}}(|t|^{3\varepsilon}+1)$$
$$f_y(2)=\sum_{d>y}\frac{\mu(d)}{d^2}\ll y^{-\frac{3}{2}+\varepsilon}, \qquad f_y{}'(2)\ll y^{-\frac{3}{2}+\varepsilon} \quad (19)$$

Then we take together (16)–(18) and put $y = x^{\frac{1-2\theta}{3-2\theta}}, T = x, c = 1+\varepsilon$,
$D^*(x) = \frac{x}{\zeta(2)}(\log x + 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}) + O(x^{\frac{1}{3-2\theta}+\varepsilon})$.

The theorem is proved.

We denote $D^*(x,m) = \sum_{\substack{n\le x\\(n,m)=1}} \tau_2^*(n)$. It is obvious that for $\operatorname{Re} s > 1$

$\sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{\tau_2^*(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} E(s,m)$, where $E(s,m) = \prod_{p/m} \frac{1+p^{-s}}{1-p^{-s}}$. Let $m = p^a$, *p* is prime , *a* is integer positive . Then for every $N \le [\frac{\log x}{\log p}]$

$$\sum_{\substack{n\le x\\(n,p)=1}} \tau_2^*(n) = D^*(x) + 2\sum_{k=1}^{N-1} D^*(\frac{x}{p^k}) + O(D^*(\frac{x}{p^N})). \qquad (20)$$

Whence we obtain

$$\sum_{\substack{n\le x\\(n,m)=1}} \tau_2^*(n) = \frac{x}{\zeta(2)}(\log x + 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)})E(1,m) + O(\Delta^*(x)\tau(m)) \quad (21)$$

while $\frac{1}{(\log\log m)^2} \le \left(\frac{\phi(m)}{m}\right)^2 \le E(1,m) = \prod_{p/m} \frac{\left(1-\frac{1}{p}\right)^2}{1-\frac{1}{p^2}} \le \left(\frac{\phi(m)}{m}\right)^2 \frac{\pi^2}{6} \le \frac{\pi^2}{6}$. The statement is proved. Now we prove the following theorem.

Theorem 2. *Let* $x \to \infty$. *The following asymptotic formula*

$$\sum_{n\le x} \tau_2^*(n)\omega(n) = c_0 x\log x\log\log x + c_1 x\log x + c_2 x\log\log x + c_3 x + O(\frac{x}{\log\log x}) \quad (22)$$

(with calculated constants $c_i, i = 0,1,2,3$ *and absolute constant in symbol O) holds.*

Proof. Let $n = p_1^{a_1}\cdot\ldots\cdot p_r^{a_r}$ is the canonical expansion of *n* . Without loss of generality we can consider that $p_1 < p_2 < \ldots < p_r, p_r = \omega(n)$. We have $n = p_1^{a_1}\cdot m_1 = p_2^{a_2}\cdot m_2 = \ldots = p_r^{a_r}\cdot m_r, (m_i, p_i) - 1, i = 1,\ldots,r$. Whence , taking into account (20), (21) we get

$$\sum_{n\le x}\tau_2^*(n)\omega(n)=\sum_{\substack{k,p\\p^k\le x}}\sum_{\substack{m\le x/p^k\\(m,p)=1}}\tau_2^*(p^k m)=\sum_{\substack{k,p\\p^k\le x}}D^*(\frac{x}{p^k},p)=$$

$$=\sum_{p\le x}\sum_{k=1}^{[\log x/\log p]}\{D^*(\frac{x}{p^k})+2\sum_{1\le j\le[\log x/\log p]-k-1}D^*(\frac{x}{p^{k+j}})\}= \tag{23}$$

$$\frac{1}{\zeta(2)}\sum_{p\le x}\{\sum_{1\le k\le\log x/\log p}[(\frac{x}{p^k}\log\frac{x}{p^k}+2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)})\frac{x}{p^k}+\Delta^*(\frac{x}{p^k}))+$$

$$+2\sum_{1\le j\le\frac{\log x}{\log p}-k-1}(\frac{x}{p^{k+j}}\log\frac{x}{p^{k+j}}(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)})+\Delta^*(\frac{x}{p^{k+j}})]+$$

$$+O(\frac{x}{p^{[\log x/\log p]}}\log x)\}$$

Applying classical Mertens results $\sum_{p\le x}\frac{1}{p}=\log\log x+c_1++O(\log^{-2}x)$, $\sum_{p\le x}\frac{\log p}{p}=\log x-c_2+O(\log^{-1}x)$, where $c_1=\gamma+\sum_p(\log(1-\frac{1}{p})+\frac{1}{p})$, $c_2=\gamma+\sum_p\frac{\log p}{p(p-1)}$, we can find the constants $c_i, i=0,1,2,3$ such that $c_0=\frac{12}{\pi^2}$,

$$\sum_{n\le x}\tau_2^*(n)\omega(n)=c_0 x\log x\log\log x+c_1 x\log x+c_2 x\log\log x+c_3 x+O(\frac{x}{\log\log x}).$$

Theorem 3. *Let* $k\ge 3$, $x\to\infty$

$$\sum_{n\le x}\tau_k^*(n)\omega(n)=\begin{cases}xQ_{k-1}(\log x)\log\log x+xR_{k-1}(\log x)+O(x),\\ \text{if \_} k=3\\ xQ_{k-1}(\log x)\log\log x+xR_{k-1}(\log x)+O(x(\log x)^{(k^2-6k+3)/(k-3)})\end{cases},$$

if $k\ge 4$ *where* $Q_{k-1}(n),R_{k-1}(n)$ *are polynomials of* $k-1$ *degree with absolute coefficients* .

Proof **.** As was shown in case $k=2$, we have

$$\sum_{n\le x}\tau_k^*(n)\omega(n)=\sum_{p^\alpha}\sum_{\substack{m\le x/p^k\\(m,p)=1}}\tau_k^*(p^\alpha m)=k\sum_{p^\alpha\le x}D_k^*(\frac{x}{p^\alpha},p), \tag{24}$$

where $D_k^*(X,p)=\sum_{\substack{n\le X\\(n,p)=1}}\tau_k^*(n), X=\frac{x}{p^\alpha}$. We denote

$F^{*}_{k,p}(s)=\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{\tau^{*}_{k}(n)}{n^{s}}=\frac{\zeta^{k}(s)}{(\zeta(2s))^{k(k-1)/2}}G_{k}(s)E^{(p)}_{k}(s)$ where $G_{k}(s)$ is regular in the domain $\operatorname{Re}s>\frac{1}{3}$; $E^{(p)}_{k}(s)=(1+\frac{k}{p^{s}}\cdot\frac{1}{1-p^{-s}})^{-1}=(1+\frac{k}{p^{s}-1})^{-1}$.

Application of Perron formula gives

$$D^{*}_{k}(X,p)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E^{(p)}_{k}(s)\frac{X^{s}}{s}ds+O(\frac{x^{c}}{p^{c\alpha T}(c-1)^{k}}) \quad (25)$$

If $k=3,4$ we remove the contour of integration on the straight line $\operatorname{Re}s=1/2+(\log x)^{-1}:=c_{0}$. We have

$$D^{*}_{k}(X,p)=\frac{1}{2\pi i}\operatorname*{res}_{s=1}(\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E_{k}(s)\frac{X^{s}}{s})+$$
$$\frac{1}{2\pi i}(\int_{c_0-iT}^{c_0+iT}+\int_{c_0+iT}^{c+iT}-\int_{c_0-iT}^{c-iT})(\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E_{k}(s)\frac{X^{s}}{s}ds)+O(\frac{x^{c}}{p^{c\alpha T}(c-1)^{k}}) \quad (26)$$

The greatest contribution into estimation of integrals in the right- hand side of (26) gives the first integral. As follows from lemmas 1,2 , for $T\le x$ we have

$$D^{*}_{k}(X,p)=\frac{1}{2\pi i}\left|\int_{c_0-iT}^{c_0+iT}\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E^{(p)}_{k}(s)\frac{X^{s}}{s}ds\right|\le$$
$$2\max_{\substack{1\le T_1\le\frac{1}{2}T\\ \operatorname{Re}s=c_0}}\{\left|\frac{G_{k}(s)E^{(p)}_{k}(s)X^{s}}{(\zeta(2s))^{k(k-1)/2}T_{1}}\right|\int_{T_1}^{2T_1}\left|\zeta^{4}(s)\right|dt\}\log T<<X^{c_0}(\log T)^{k(k-1)/3}\log^{5}T<<$$
$$X^{1/2}(\log T)^{9} \quad (27)$$

Besides for $k=3,4,5,..$

$$\operatorname*{res}_{s=1}(\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E_{k}(s)\frac{X^{s}}{s}=xP_{k-1}(\log x), \quad (28)$$

where $P_{k-1}(n)$ is the polynomial of *k*–1 degree , which coefficients $c_{ik}=d_{ik}+O(\frac{1}{p}),c_{k-1,k}=\frac{p-1}{p+k-1}(\frac{\pi^{2}}{6})^{k(k-1)/2}G_{k}(1)>0,d_{ik}$ are absolute constants.

For $k\ge5$ we apply the estimation of *k*-th moment of $\zeta(s)$ on the middle straight line (see lemma 2)

$$\int_{1}^{T}\left|\zeta(\frac{1}{2}+it)\right|^{k}dt<<T^{m(k)+\varepsilon}$$ and so we have

$$\frac{1}{2\pi i}\int_{c_0-iT}^{c_0+iT}\zeta^{k}(s)(\zeta(2s))^{-k(k-1)/2}G_{k}(s)E^{(p)}_{k}(s)\frac{X^{s}}{s}ds<<X^{1/2}(\log T)^{k(k-1)/3}T^{m(k)+\varepsilon}(1+\frac{k}{\sqrt{p}})$$

Moreover,

$$I_1, I_2 \ll \int_{c_0}^{c} \left| \frac{\zeta(\sigma \pm iT)}{(\zeta(2\sigma \pm 2iT))^{k(k-1)/2}} \right| \left| G_k(\sigma \pm iT \right| \left| E_k^{(p)}(\sigma \pm iT) \right| \frac{X^\sigma}{T} d\sigma \ll$$

$$\frac{(\log T)^{k(k-1)/3}}{T} (1 + O(\frac{k}{\sqrt{p}})) \int_{c_0}^{c} (T^{(1-\sigma)/3})^k X^\sigma d\sigma \ll$$

$$(1 + \frac{k}{\sqrt{p}})(\log T)^{k(k-1)/3} T^{\frac{k}{3}-1} \int_{c_0}^{c} \left(\frac{X}{T^{k/3}} \right)^\sigma d\sigma \ll (1 + \frac{k}{\sqrt{p}})(\log T)^{k(k-1)/3} \cdot$$

$$\cdot T^{\frac{k}{3}-1} (X^{1/2} T^{-k/6} + X T^{-k/3})$$

Then

$$D^*(X, p) = X P_{k-1}(\log x) + O(X^{1/2} T^{m(k)+\varepsilon} (1 + \frac{k}{\sqrt{p}})(\log T)^{k(k-1)/2}) +$$

$$+ O((1 + \frac{k}{\sqrt{p}}) \frac{X}{T} (\log T)^{k(k-1)/3}).$$

Noting that $X = \frac{x}{p^\alpha}$ and summing over all prime *p* we obviously obtain

$$\sum_{n \le x} \tau_k^*(n)\omega(n) = x Q_{k-1}(\log x) \log\log x + x R_{k-1}(\log x) + O(\frac{x}{\log x} T^{\frac{k}{3}-\frac{7}{6}} (\log x)^{k(k-1)/2}) +$$

$$+ O(x \log\log x T^{\frac{k-4}{3}+\varepsilon} (\log T)^{k(k-1)/3}) + O(x^{1/2} \log\log x T^{\frac{k-4}{3}-\frac{1}{6}}) + O(k x T^\varepsilon)$$

For $k = 3$ putting $T = (\log x)^{k/\log x}$ and for $k \ge 4$ $T = (\log x)^{3(k-1)/(k-3)}$ we obtain the statement of the theorem

$$\sum_{n \le x} \tau_k^*(n)\omega(n) = \begin{cases} x Q_{k-1}(\log x) \log\log x + x R_{k-1}(\log x) + O(x), \\ \text{if\_} k = 3 \\ x Q_{k-1}(\log x) \log\log x + x R_{k-1}(\log x) + O(x (\log x)^{(k^2-6k+3)/(k-3)}) \end{cases},$$

if $k \ge 4$ with the constants in symbols O depending on *k* only.

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