ON THE DISTRIBUTION OF ZEROS OF FUNCTIONS OF BERGMAN SPACE

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Basic definitions and notations. Let $D_r = \{z \in \mathbb{C} : |z| < r\}$, $D = D_1; d_r(\zeta) = \{z \in \mathbb{C} : |z - \zeta| < r\}$, $T_r = \{z \in \mathbb{C} : |z| = r\}$, $T = T_1, t_r(\zeta) = \overline{d}_r(\zeta) \setminus d_r(\zeta)$, let $d\sigma$ be the planar Lebesgue measure; let the space $H'_p (0 < p < \infty)$ be the set of analytic in the area D functions, belonging to the $L_p(D)$ space. Let the space $H_p (0 < p \le \infty)$ be the set of functions, which are analytic ones in the area D with the finite quantity $\lim\limits_{r\to 1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p}$.

M.M. Djrbashian seems to be the first who began to study the spaces H'_p in the paper [1]; the symbol H'_2 was used by A.I. Markushevich in the theorem on the approximation in the mean of analytic functions with polynomials in [2] , in the book [3] the sufficiently complete information of Bergman spaces is given. In the papers [1], [3], [4], [5], [6] and [7] different sufficient conditions distribution of zeros of functions of the space H'_p are studying. In [5] it is pointed out, that the criterion of distribution of zeros considerably depends on the p value. In this paper the necessary and sufficient condition of zeros distributing in the unit circle of functions from H'_p is obtained.

The main results of the published paper are Theorem 1, which is the necessary and sufficient condition of the distribution of zeros of functions of the space H'_p, and the theorem on representation of functions from H'_p as a product of two factors, one of which doesn't become zero in the unit circle.

Theorem 1.

1. *If* $\{a_j\}, 0 < |a_1| \le |a_2| \le \ldots \le |a_n| \le \ldots < 1$, *is a sequence of zeros of a function* $f \in H'_p, 0 < p < \infty$, *then*

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}\frac{1}{|a_k|^p}\frac{|a_{n+1}|^{np+1} - |a_n|^{np+1}}{np+1} < \infty \qquad (1)$$

2. *If* $\{a_j\}, 0 < |a_1| \le |a_2| \le \ldots \le |a_n| \le \ldots < 1$, *is a sequence of zeros of some function* f *, which is analytic in* D, *and the series (1) diverges, then* $f \notin H'_p$.

Proof. From the Poisson–Jensen formula ([8], p. 165), considering that $|f(0)|^{-1/p} = C > 0$, the result

$$\sum_{|a_k|<r} \ln \frac{r}{|a_k|} = -\ln|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \qquad (2)$$

is deduced.

Multiplying by the p both parts of (2) and then exponentiating, we obtain

$$\prod_{|a_k|<r} \left(\frac{r}{|a_k|}\right)^p = C\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|^p d\theta\right). \qquad (3)$$

Then acting like ([9] p.111), we deduce from (3)

$$\prod_{|a_k|<r} \left(\frac{r}{|a_k|}\right)^p \le C\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta .$$

Integrating the obtained equality over $r, |a_n| \le r < |a_{n+1}|, n = 1, 2, ...$

$$\prod_{|a_k|<|a_{n+1}|} \left(\frac{1}{|a_k|}\right)^p \frac{|a_{n+1}|^{pn+1} - |a_n|^{pn+1}}{pn+1} \le C\frac{1}{2\pi}\int_{|a_n|}^{|a_{n+1}|}\int_0^{2\pi} |f(re^{i\theta})|^p rdrd\theta$$ and summing up, we obtain

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n} \left(\frac{1}{|a_k|}\right)^p \frac{|a_{n+1}|^{pn+1} - |a_n|^{pn+1}}{pn+1} \le C\frac{1}{2\pi}\int_{|a_1|}^{1}\int_0^{2\pi} |f(re^{i\theta})|^p rdrd\theta \qquad (4)$$

which implies that under the condition $f \in H'_p$ the series from (4) converges, and therefore the series from (1) converges and vice versa, if the series from (1) diverges, then the series from (4) also diverges, and consequently $f \notin H'_p$.

Theorem1 is proved.

Corollary. *If* $\sum_{n=1}^{\infty}\prod_{k=1}^{n} \left(\frac{|a_n|}{|a_k|}\right)^p (|a_{n+1}| - |a_n|) = \infty$ *then* $f \notin H'_p$, *and if* $f \in H'_p$ *then* $\sum_{n=1}^{\infty}\prod_{k=1}^{n} \left(\frac{|a_{n+1}|}{|a_k|}\right)^p (|a_{n+1}| - |a_n|) < \infty$.

Proof. From the equality $\beta^\mu - \alpha^\mu = \mu\gamma^{\mu-1}(\beta-\alpha)$, $0 \le \alpha < \gamma < \beta, \mu > 1$, an inequality

$$\frac{p}{p+1}|a_n|^n (|a_{n+1}| - |a_n|) \le \frac{|a_{n+1}|^{np+1} - |a_n|^{np+1}}{np+1} \le |a_{n+1}|^n (|a_{n+1}| - |a_n|) \qquad (5)$$

follows, and the statement of the corollary is obtained from (5) on the basis of Theorem 1. Proof is completed.

Let a_j be the sequence of points in D. Let $b_{n,r}(z)$ be the Blaschke function in the circle of radius r:

$$b_{n,r}(z) = \left(\frac{z}{r}\right)^m \prod_{k=1}^{n} \frac{r|a_k|}{a_k} \frac{a_k - z}{r^2 - \bar{a}_k z}, |a_k| \le r \qquad (6)$$

In the product each factor is repeated as many times as the multiplicity of a_j is.

Convention. In the future, if it is inessential for this proof, all a_j, which have equal absolute values, will have the same index, otherwise $|a_j|e^{i\alpha}$ will be written down, where α sequentially takes all values of $\arg a_j$.

Assume that $B_{n,r}(z) = b_{n,r}(z),\ |a_n| \le r < |a_{n+1}|,$ and $B_{n,|a_n|}(z) = b_{n,|a_n|}(z),$ $|z| = |a_n|, z \ne |a_n|e^{i\alpha}$.

Lemma 1. *Function*

$$\mathrm{B}(\varsigma) = \frac{1}{2\pi}\int_0^{2\pi} B_{n,r}(\zeta + \rho_\varsigma e^{i\theta})d\theta, \zeta = re^{i\theta}, |a_n| \le r < |a_{n+1}| \qquad (7)$$

is continuous in D.

Proof. Let us choose $\rho_\varsigma = \min(|\varsigma| - |a_n|, |a_{n+1}| - |\varsigma|)$, $|a_n| < |\varsigma| < |a_{n+1}|$, and if $|\varsigma = |a_n|e^{i\phi}, \phi \ne \alpha,$, we assume that $\mathrm{B}(\varsigma) = \lim\limits_{\rho_\varsigma \to 0} \frac{1}{2\pi}\int_0^{2\pi} B_n(\zeta + \rho_\varsigma e^{i\theta})d\theta$.

Function B is continuous in the ring $|a_{n-1}| \le |z| \le |a_n|$, however, in $T_{|a_n|}$ it can be defined both with $B_{n-1,r}$ and with $B_{n,r}$, and the first one coincides there with $b_{n-1,r}$, while the second one coincides with $b_{n,r}$.

Note, that $b_{n-1,|a_n|}(|a_n|e^{i\theta}) = \left(\frac{|a_n|e^{i\theta}}{|a_n|}\right)^m \prod\limits_{k=1}^{n-1} \frac{|a_n||a_k|}{a_k} \frac{a_k - |a_n|e^{i\theta}}{|a_n|^2 - \bar{a}_k|a_n|e^{i\theta}}$,

and

$$b_{n,|a_n|}(|a_n|e^{i\theta}) = \frac{|a_n|^2}{a_n}\frac{a_n - |a_n|e^{i\theta}}{|a_n|^2 - \bar{a}_n|a_n|e^{i\theta}} b_{n-1,|a_n|}(|a_n|e^{i\theta}) = \frac{1 - e^{i(\theta-\alpha)}}{1 - e^{i(\theta-\alpha)}} b_{n-1,|a_n|}(|a_n|e^{i\theta})$$

Consequently, B is continuous in D, with the exception of points a_j, which are considered to be points with removable singularities. Proof is completed.

Lemma 2. *Cauchy–Riemann conditions for the function* B *are satisfied in* $\mathrm{D} \setminus \{a_j\}$.

Proof. Introduce operators: $\frac{\partial}{\partial z} = 1/2\left(e^{-i\theta}\frac{\partial}{\partial r} - ie^{-i\theta}\frac{\partial}{\partial \theta}\right)$, $\frac{\partial}{\partial \bar{z}} = 1/2\left(e^{-i\theta}\frac{\partial}{\partial r} + ie^{-i\theta}\frac{\partial}{\partial \theta}\right)$, $z = re^{i\theta}$.

It can be asserted that in a certain point the first operator exists, and the second one is zero, then this point satisfies Cauchy–Riemann conditions. The converse is true.

Note (7) leads to $\dfrac{\partial B(\zeta)}{\partial\zeta} = \dfrac{1}{2\pi}\int\limits_{t_{\rho_\zeta}} (B_{n,|\zeta|}(\zeta+\rho e^{i\varphi}))'_\zeta\, d\varphi = B'_{n,|\zeta|}(\zeta)$, but $\dfrac{\partial B(\zeta)}{\partial\bar\zeta} = \dfrac{1}{2\pi}\int\limits_{t_{\rho(\zeta)}} (B_{n,|\zeta|}(\zeta+\rho e^{i\varphi}))'_{\bar\zeta}\, d\varphi = 0$. Consequently, Cauchy–Riemann conditions for B are satisfied in the ring $|a_{n-1}|\le|\zeta|\le|a_n|$, excepting only points $|a_n|e^{i\alpha}$.

Prove the possibility of "gluing" on $T_{|a_n|}\setminus\{|a_n|e^{i\alpha}\}$. We have $\dfrac{\partial B(|a_n|e^{i\theta})}{\partial\zeta} = B'_{n,|a_n|}(\zeta)|_{\zeta=|a_n|e^{i\theta}} = B'_{n-1,|a_n|}(\zeta)|_{\zeta=|a_n|e^{i\theta}}\dfrac{1-e^{i(\theta-\alpha)}}{1-e^{i(\theta-\alpha)}} +$
$B_{n-1,|a_n|}(\zeta)|_{\zeta=|a_n|e^{i\theta}}\dfrac{|a_n|^2}{a_n}\left(\dfrac{a_n-\zeta}{|a_n|^2-\bar a_n\zeta}\right)'_{\zeta=|a_n|e^{i\theta}} = B'_{n-1,|a_n|}(\zeta)|_{\zeta=|a_n|e^{i\theta}} =$
$= \dfrac{\partial B(|a_n|e^{i\theta})}{\partial\zeta}, \theta\ne\alpha$. Proof is completed.

Lemma 3. *Let the function be* $\phi\in H'_1(H_1)$, *and* $\phi(a_k)=0, a_k\in D$. *Then* $(0<p<\infty)$ $H(z)=\dfrac{\phi(z)}{B(z)}$ *is continuous in* D *and satisfies Cauchy–Riemann conditions in* $D\setminus\{a_k\}$, *and it satisfies the following estimates*

$$\int_0^{2\pi} |H(re^{it})|^p\, dt \le \|\phi\|^p_{H_p} \tag{8}$$

$$\iint_D |H(re^{it})|^p\, r dr dt \le \|\phi\|^p_{H'_p} \tag{9}$$

Proof. Let $\phi(z)$ be an analytic function in $\bar D$, its zeros $a_k, k=1,2,...,N$ belong to the circle D. Assume that $H(z)=\dfrac{\phi(z)}{B(z)}, z\in D$. As follows from the analyticity of ϕ in D and properties of function B, which were determined by Lemma 1 and Lemma 2, $H(z)$ is continuous in D and in the points a_k aren't zero in D; it satisfies Cauchy–Riemann conditions everywhere in D with the exception of the points a_k. Note also, that because of the subharmonicity of ϕ the inequalities are satisfied

$$\int_0^{2\pi} |H(re^{it})|^p\, dt \le \int_0^{2\pi} |\phi(re^{it})|^p\, dt \le \|\phi\|^p_{H_p} \tag{10}$$

$$\iint_{D_R} |H(re^{it})|^p\, r dr dt \le \iint_{D_R} |\phi(re^{it})|^p\, r dr dt \le \|\phi\|^p_{H'_p} \tag{11}$$

Lemma 3 is proved.

Theorem 2. *If* $\phi \in H'_p, 0 < p < \infty, \phi(a_k) = 0$, *then it can be written as* $\phi(z) = \Phi(z)\Psi(z), \Phi(z) = \left(H(z)\right)^{1/2} \neq 0, z \in D, \Psi(z) = \left(H(z)\right)^{1/2} B(z), z \in D$, $\Phi, \Psi \in H_{2p}'$ *and*

$$\| \Phi \|_{H'_2}^{2p} \leq \| \phi \|_{H'_p}^{1/2}, \| \Psi \|_{H'_{2p}}^{2p} \leq \| \phi \|_{H'_p}^{1/2} . \tag{12}$$

Proof. We will use Theorem 1 from [10].

Let the function $f(z) = u(x, y) + iv(x, y), z = x + iy$, be certain in the domain J and have the following properties:

1) $u(x, y), v(x, y)$ have partial derivatives u'_x, u'_y, v'_x, v'_y (or finite partial Dini derivatives) everywhere in J, except, possibly, a set which is the sum of a finite or countable number of closed sets of finite one-dimensional Hausdorff measure;
2) $u(x, y), v(x, y)$ satisfy Cauchy–Riemann conditions almost everywhere in J;
3) $u(x, y), v(x, y)$ are linearly continuous (on x and y);
4) $| f(z) |$ is a locally summable function.

Then $f(z)$ is an analytic function in J.

From Lemmas 2 and 3 we conclude, that Φ and Ψ are analytic in $D \setminus \{a_k\}$ and continuous in D (Lemmas 1 and 3), consequently, the conditions of Theorem 1 from [10] are satisfied, therefore the functions Φ and Ψ are analytic in D, and $\Phi(z) \neq 0, z \in D$ because $H(z)$ is the same in D. In this case, the inequality (9) entails

$$\| \Phi \|_{H'_{2p}} = \left(\| H \|_{H_p} \right)^{1/2} \leq \left(\| f \|_{H'_p} \right)^{1/2} \tag{13}$$

$$\| \Psi \|_{H'_{2p}} = \left(\| HB \|_{H_p} \right)^{1/2} \leq \left(\| f \|_{H'_p} \right)^{1/2} \tag{14}$$

Theorem 2 is proved.

Theorem 3. If $f \in H'_p, 0 < p < \infty$, then

$$f(z) = f_+(z) + f_-(z), f_\pm \in H'_p, f_\pm(z) \neq 0, z \in D, \tag{15}$$

$$\| f_\pm \|_p \leq \| \phi \|_p \tag{16}$$

Proof. Assume that $f_+ = \dfrac{H(1+B)}{2}$ and $f_- = -\dfrac{H(1-B)}{2}$, and the validity of (15) follows from the previously proved lemmas, concerning the inequality (16), it follows from (9). Theorem 3 is proved.

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