MATHEMATICAL FORMULATIONS AND NUMERICAL SOLUTIONS OF INITIAL-BOUNDARY-VALUE PROBLEM OF CREEP THEORY

Romashov Yu.V., Sobol V.N.

National technical university "Kharkov polytechnic institute", Ukraine

Initial-boundary problem for the creep theory of deformable solid describes stress–strain state taking into account the non-stationary irreversible creep strains and material damage. Creep problems are actual for such engineering systems that improving of their operating characteristics can be provided by increasing temperature and mechanical stresses (working pressure, rotating frequency, etc.) [1, 2].

Possibility of analytical solution for problems of creep theory is limited by equations nonlinearity and transient nature of unknown fields. At such case a numerical solutions can be used. Usually creep problems statements in displacements are given. For their numerical solution finite element method [3] and the variational–structural method of R–functions theory [4] have been proposed. Improvement of the state equations, for example by introduction of additional state parameters [5], leads to improving of numerical methods for solving problems of creep theory. Further, the mathematical formulation and algorithmization of numerical solution for initial-boundary-value problems of creep theory are considered, that almost haven't discussion in the literature. So it is desirable to develop efficient software [6].

1. Mathematical formulations of initial-boundary-value creep problems. Problems of creep theory consider for a body with volume Y and the boundary surface υ. Let's consider a case of small deformations and static loads, which often used in engineering applications. Complete strain tensor consists of a sum of reversible elastic strain and irreversible creep strain tensors; Hooke's law can be used for describing of elastic strains, a body material – homogeneous and isotropic. Temperature field at body points accepted as homogeneous and stationary. Under these assumptions, the problem formulation of creep theory includes a description of the stress–strain state in elastic deformation, taking into account creep strains and law description of the creep strains changes over a time. Body points individualizing at initial state in compliance with Lagrange method, which corresponds to the time $t = 0$, and consider at the moments $t \geq 0$ [7]. Cartesian coordinates x_k are used in the paper. All subscripts have the values 1, 2, 3, if their values are not specified additionally. Einstein summation convention is used.

Stress–strain state of a body by taking to account creep strains can be written by using equations of deformable solid mechanics and corresponding boundary conditions:

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right), \sigma_{ij,j} + f_i = 0, \varepsilon_{ij} = e_{ij} + c_{ij}, e_{ij} = a_{ijkl}\sigma_{kl} \ \forall x_k \in Y , \qquad (1)$$

$$u_i = \widetilde{u}_i \ \ \forall x_k \in \upsilon_u, \quad \sigma_{ij} n_j = p_i \ \ \forall x_k \in \upsilon_p, \tag{2}$$

where $u_{i,j} \equiv \partial u_i / \partial x_j$ and etc.; $u_i = u_i(x_k, t)$ and $\sigma_{ij} = \sigma_{ij}(x_k, t)$ – displacements and stress components; $\varepsilon_{ij} = \varepsilon_{ij}(x_k, t)$, $e_{ij} = e_{ij}(x_k, t)$ and $c_{ij} = c_{ij}(x_k, t)$ – strain components: full, reversible elastic and irreversible creep; $f_i = f_i(x_k)$ – specified components of a volume force, $\widetilde{u}_i = \widetilde{u}_i(x_k)$ – specified components of displacements at surface points $\upsilon_u \subset \upsilon$; $p_i = p_i(x_k)$ – specified components of surface loads at surface points $\upsilon_p \subset \upsilon$; n_j – components of external unit normal to a body surface; a_{ijkl} – elastic constants of a body material.

Problem formulation (1), (2) about strain–stress state under creep process is responding to a full equation system in full state space [8]. Resolving equations can be considered in partial state space by excluding unknowns [8]. For example, equations in displacements and stresses can be considered with boundary conditions (2) except of equations (1)

$$-2a_{ijkl}\sigma_{kl} + u_{i,j} + u_{j,i} = 2c_{ij}, \sigma_{ij,j} + f_i = 0 \ \forall x_k \in Y. \tag{3}$$

Also resolving equations can be considered in displacements

$$a^{-1}_{ijkl}\left(u_{k,lj} + u_{l,kj}\right) = -2 f_i + 2a^{-1}_{ijkl} c_{kl,j} \ \forall x_k \in Y, \tag{4}$$

for which the second boundary condition (2) becomes

$$\left[a^{-1}_{ijkl}\left(u_{k,lj} + u_{l,kj}\right) - 2a^{-1}_{ijkl} c_{kl}\right] n_j = 2 p_i \ \forall x_k \in \upsilon_p. \tag{5}$$

Here $a^{-1}_{ijkl} a_{klmq} = \delta_{im}\delta_{jq}$, δ_{ij} – Kronecker delta. Advantage of the partial states spaces is a smaller number of unknowns. Disadvantage of some partial state spaces – more complicated form of boundary conditions due to the presence of creep strains. For example, the boundary conditions (5) for equations (4) have a creep strains, but the boundary conditions (2) for resolving equations (3) is considered without it. Next we consider formulations only for state spaces with boundary conditions without creep strains.

Variational problem formulations of stress–strain state for a body during the creep process are used except of local formulations. Functional variation must be written for given creep strains c_{ij}. As, for example, variational formulation on the basis of mixed Reissner functional can be written [9]:

$$J_R(u_i, \sigma_{ij}; c_{ij}) = \int_Y \left[\tfrac{1}{2}\sigma_{ij}\left(u_{i,j} + u_{j,i}\right) - \tfrac{1}{2} a^{-1}_{ijkl}\sigma_{ij}\sigma_{kl} - u_i f_i\right] dY - \int_{\upsilon_p} u_i p_i d\upsilon -$$

$$- \int_{\upsilon_u} \sigma_{ij} n_j \left(u_i - \widetilde{u}_i\right) d\upsilon - \int_Y \sigma_{ij} c_{ij} dY, \tag{6}$$

$$\delta J_R(u_i, \sigma_{ij}; c_{ij}) = \int_Y \left[\left(\tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) - a^{-1}_{ijkl}\sigma_{kl} - c_{ij}\right)\delta\sigma_{ij} - \left(\sigma_{ij,j} + f_i\right)\delta u_i\right] dY -$$

$$- \int_{\upsilon_p} \left(\sigma_{ij} n_j + p_i\right) \delta u_i d\upsilon - \int_{\upsilon_u} \delta\sigma_{ij} n_j \left(u_i - \widetilde{u}_i\right) d\upsilon . \qquad (7)$$

Non-stationary character of stress–strain state can be explained by creep strains changing over a time. Further generalized creep law with initial conditions is used [10]:

$$\dot{c}_{ij} = \frac{3}{2} \frac{\dot{c}}{\sigma^{(c)}} s_{ij}, \; s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \quad c_{ij}\left(x_k, 0\right) = 0 \quad \forall x_k \in Y , \qquad (8)$$

where $\dot{c}_{ij} \equiv \partial c_{ij} / \partial t$; $\sigma^{(c)} = \sigma^{(c)}\left(\sigma_{ij}\right)$ – equivalent stresses for creep strain rate controlling; $\dot{c}$ – intensity of creep strains rate.

Intensity of creep strain rate can be given in the form of Bailey–Norton law:

$$\dot{c} = B\left(\sigma^{(c)}\right)^n . \qquad (9)$$

Here B, n – material characteristics for given temperature, it can be found from experimental creep curves. Influence of forming and developing micropores on creep strain rate can be taken into account by inserting of scalar damage parameter $\omega = \omega\left(x_k, t\right)$ [2, 9, 11]:

$$\dot{c} = B\left(\frac{\sigma^{(c)}}{1-\omega}\right)^n , \quad \dot{\omega} = A\left(\frac{\sigma^{(\omega)}}{1-\omega}\right)^m , \quad \omega\left(x_k, 0\right) = 0 \qquad \forall x_k \in Y , \qquad (10)$$

where A, m – material characteristics for given temperature, which can be taken from experimental creep rupture strength curves; $\sigma^{(\omega)} = \sigma^{(\omega)}\left(\sigma_{ij}\right)$ – equivalent stresses for creep strain rate controlling.

Problem of creep theory is considered for full space as nonlinear boundary-initial-value problem (1), (2), (8), (9) or (10). Time integration is performed until phase ending of nonlocalized rupture and localized defect formation (crack), that determined by dependence

$$\omega\left(x_k^*, t^*\right) = 1 . \qquad (11)$$

where t^* – formation time and x_k^* – coordinates of localized defect, for time moments $t > t^*$ body behavior can't be described adequately by equations (1), (2), (8), (9) and (10). Because they don't take into account localized defects, for example cracks.

2. Numerical solution of initial-boundary-value problems of creep theory. Let's introduce the vectors $\mathbf{u}^{(1)} = \mathbf{u}^{(1)}\left(x_k, t\right) \in R^N$, $N = N^{(1)}$ – with stress–strain state parameters, and $\mathbf{u}^{(2)} = \mathbf{u}^{(2)}\left(x_k, t\right) \in R^N$, $N = N^{(2)}$ – with creep strains and damage parameter. Stress–strain state problem for elastic deformation of a body with creep strains including can be written in local form:

$$\mathbf{A}^{(1)} \cdot \mathbf{u}^{(1)} + \mathbf{A}^{(2)} \cdot \mathbf{u}^{(2)} = \mathbf{f}^{(1)} \; \forall x_k \in Y, \quad \mathbf{L} \cdot \mathbf{u}^{(1)} = \widetilde{\mathbf{u}}^{(1)} \; \forall x_k \in \upsilon . \qquad (12)$$

Here $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ – linear operators and $\mathbf{f}^{(1)}$ – vector, corresponding to the differential equations, a $\mathbf{L}$ – linear operator and $\tilde{\mathbf{u}}^{(1)}$ – vector, corresponding to the boundary conditions. Variational problem statement can be presented by analogy (7) in the form of variation, that equal to zero

$$\int_Y \left(\delta\mathbf{u}^{(1)}\right)^T \cdot \left(\mathbf{A}^{(1)} \cdot \mathbf{u}^{(1)} + \mathbf{A}^{(2)} \cdot \mathbf{u}^{(2)} - \mathbf{f}^{(1)}\right) dY + \int_\upsilon \delta\mathbf{u}^{(1)} \cdot \left(\mathbf{L} \cdot \mathbf{u}^{(1)} - \tilde{\mathbf{u}}^{(1)}\right) d\upsilon = 0 . \quad (13)$$

Creep law (8), (9) or (10) in operator form can be given

$$\dot{\mathbf{u}}^{(2)} = \mathbf{f}^{(2)}\left(\mathbf{u}^{(2)}; \mathbf{u}^{(1)}\right), \quad \mathbf{u}^{(2)}(x_k, 0) = \mathbf{0} \quad \forall x_k \in Y . \quad (14)$$

Here vector $\mathbf{f}^{(2)}$ corresponds to the creep law (8) with (9) or (10). Thus, problem of creep theory uniformly can be presented in local form (12), (14) and in variation form (13), (14). Such forms are independent from state space choice. Vectors and operators form (12)–(14) correspond to each state space.

Firstly let's consider the problem solution of creep theory on the basis of most common formulation in literature (13), (14). Creep strains take into account in specially selected points in solving of variational problem (13). In such way, numerical methods for problems solution of elasticity theory can be used. Further variational–structural method of R–function theory [12] is used for problem solution (13). Field of stress–strain state can be presented in the form for satisfying of the boundary conditions (12) of solutions structures

$$\mathbf{u}^{(1)} = \mathbf{B}^{(1)}\left(\mathbf{\Phi}^{(1)}, Y, \upsilon, \mathbf{L}, \tilde{\mathbf{u}}^{(1)}\right). \quad (15)$$

Undefined components $\mathbf{\Phi}^{(1)} = \mathbf{\Phi}^{(1)}(x_k)$ of solutions structures (15) are considered in the basis function approximation of space coordinates [12]

$$\mathbf{\Phi}^{(1)}(x_k) = \sum_{i=1}^{n} \boldsymbol{\varphi}_i^{(1)}(x_k) \cdot a_i^{(1)} . \quad (16)$$

Here n – number of basis functions; $a_i^{(1)}$ – approximation coefficients; $\boldsymbol{\varphi}_i^{(1)}(x_k)$ – basis functions, that are included as elements of functional space with $\mathbf{\Phi}^{(1)}$ [12]. Substitution of approximations (16) in solutions structures (15) lead to expressions in the form of linear combinations of approximation coefficients [12] in the case of linear operators for boundary conditions. It can be written in matrix–vector form as

$$\mathbf{u}^{(1)}(x_k) = \mathbf{u}_\upsilon^{(1)}(x_k) + \mathbf{U}_n^{(1)}(x_k) \cdot \mathbf{a}_n^{(1)}, \quad \delta\mathbf{u}^{(1)}(x_k) = \mathbf{U}_n^{(1)}(x_k) \cdot \delta\mathbf{a}_n^{(1)}, \quad (17)$$

where $\mathbf{u}_\upsilon^{(1)}$ – vector for extending of boundary values inside a body region; $\mathbf{U}_n^{(1)}$ – matrix of trial functions, composed from multiplier in coefficients $a_i^{(1)}$; vector $\mathbf{a}_n^{(1)} = \left(a_1^{(1)} \quad a_2^{(1)} \quad \dots \quad a_n^{(1)}\right)^T$.

Substitutions of structures (17) in expression (13) with fact of variation independence $\delta \mathbf{a}_n^{(1)}$ lead to the equation system

$$\mathbf{A}_n^{(1)} \cdot \mathbf{a}_n^{(1)} = \mathbf{f}_n^{(1)} - \int_Y \left(\mathbf{U}_n^{(1)}\right)^T \cdot \left(\mathbf{A}^{(2)} \cdot \mathbf{u}^{(2)}\right) dY . \qquad (18)$$

where $\mathbf{A}_n^{(1)} = \int_Y \left(\mathbf{U}_n^{(1)}\right)^T \cdot \left(\mathbf{A}^{(1)} \cdot \mathbf{U}_n^{(1)}\right) dY$, $\mathbf{f}_n^{(1)} = \int_Y \left(\mathbf{U}_n^{(1)}\right)^T \cdot \left(\mathbf{f}^{(1)} - \mathbf{A}^{(1)} \cdot \mathbf{u}_\upsilon\right) dY$.

Equation integral (18) can be presented in the quadrature form

$$\int_Y \left(\mathbf{U}_n^{(1)}\right)^T \cdot \left(\mathbf{A}^{(2)} \cdot \mathbf{u}^{(2)}\right) dY \approx \sum_{i=1}^{Q} w^{(i)} \left(\mathbf{U}_n^{(1)}\left(\xi_k^{(i)}\right)\right)^T \cdot \left(\mathbf{A}^{(2)} \cdot \mathbf{u}^{(2)}\left(\xi_k^{(i)}\right)\right). \qquad (19)$$

Here Q – number, $w^{(i)}$ –weighting coefficients and $\xi_k^{(i)}$ – nodes coordinates of quadrature formula. Equation (14) is integrated in nodes of quadrature formula (19):

$$\dot{\mathbf{u}}^{(2)}\left(\xi_k^{(i)}\right) = \mathbf{f}^{(2)}\left(\mathbf{u}^{(2)}\left(\xi_k^{(i)}\right), \mathbf{u}^{(1)}\left(\xi_k^{(i)}\right)\right), \quad \mathbf{u}^{(2)}\left(\xi_k^{(i)}, 0\right) = \mathbf{0} \; \forall x_k \in Y, i = 1, \ldots, Q . \qquad (20)$$

Thus, the initial-boundary-value problem solution of creep theory is reduced to the integration of ordinary differential equations with initial conditions (20). Boundary-value problem is solved on the integration steps (18), (19). As a result, on the steps of time integration we have fields of stress–strain state, creep strains and damage parameter values in nodes of quadrature formulas and others predetermined points of a body. Determine ability of creep strains and damage parameter only in defined body points makes it difficult to determine the time moment t^* and localized defect coordinates x_k^*, because high accuracy requires a large number of points.

Problem solution of creep theory (12), (14) can be done by using weighted residual method [13]. Problem unknowns by using of structural method of R–function theory [12] are given in the form of identically satisfying to the boundary conditions (12) of solutions structures

$$\mathbf{u}^{(1)} = \mathbf{B}^{(1)}\left(\mathbf{\Phi}^{(1)}, Y, \upsilon, \mathbf{L}, \tilde{\mathbf{u}}^{(1)}\right), \quad \dot{\mathbf{u}}^{(2)} = \mathbf{\Phi}^{(2)}, \qquad (21)$$

with undetermined coefficients $\mathbf{\Phi}^{(1)} = \mathbf{\Phi}^{(1)}(x_k, t)$ and $\mathbf{\Phi}^{(2)} = \mathbf{\Phi}^{(2)}(x_k, t)$, in the approximations form by using of basis functions systems of space coordinates with time-dependent coefficients

$$\mathbf{\Phi}^{(1)}(x_k, t) = \sum_{i=1}^{n} \boldsymbol{\varphi}_i^{(1)}(x_k) \cdot a_i^{(1)}(t), \quad \mathbf{\Phi}^{(2)}(x_k, t) = \sum_{i=1}^{n} \boldsymbol{\varphi}_i^{(2)}(x_k) \cdot a_i^{(2)}(t) . \qquad (22)$$

Here $a_i^{(1)}(t)$ and $a_i^{(2)}(t)$ – approximation coefficients; $\boldsymbol{\varphi}_i^{(1)}(x_k)$ and $\boldsymbol{\varphi}_i^{(2)}(x_k)$ – basis functions, that are included as elements of functional space with $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$. In the case of linear operators of boundary conditions,

approximations substitution (22) in the solutions structures (21) leads to expressions as linear combinations of approximations coefficients [12], which can be written in matrix–vector form

$$\mathbf{u}^{(1)}(x_k,t)=\mathbf{u}_{\upsilon}^{(1)}(x_k)+\mathbf{U}_n^{(1)}(x_k)\cdot\mathbf{a}_n^{(1)}(t),\ \ \mathbf{u}^{(2)}(x_k,t)=\mathbf{U}_n^{(2)}(x_k)\cdot\mathbf{a}_n^{(2)}(t), \qquad (23)$$

where $\mathbf{U}_n^{(2)}$ – matrix of trial functions, composed from multipliers in coefficients $a_i^{(2)}$ in solutions structures (21) after substitutions of approximations (22); $\mathbf{a}_n^{(2)}(t)=\left(a_1^{(2)}(t)\ \ a_2^{(2)}(t)\ \ \ldots\ \ a_n^{(2)}(t)\right)^T$, $\mathbf{a}_n^{(2)}(0)=\mathbf{0}$.

Approximations (23) can be presented in matrix–vector form for determination if approximations coefficients by using weighted residuals method

$$\mathbf{u}(x_k,t)=\mathbf{u}_{\upsilon}(x_k)+\mathbf{U}_n(x_k)\cdot\mathbf{a}_n(t), \qquad (24)$$

and differential equations (12), (14) – in the form of one equation [14]

$$\mathbf{K}\cdot\dot{\mathbf{u}}+\mathbf{A}\cdot\mathbf{u}=\mathbf{f}(\mathbf{u})\ \forall t\ge 0\ \forall x_k\in Y\ . \qquad (25)$$

Here block vectors and matrices can be introduced

$$\mathbf{u}=\begin{pmatrix}\mathbf{u}^{(1)}\\ \mathbf{u}^{(2)}\end{pmatrix},\mathbf{u}_{\upsilon}=\begin{pmatrix}\mathbf{u}_{\upsilon}^{(1)}\\ \mathbf{0}\end{pmatrix},\mathbf{U}_n=\begin{pmatrix}\mathbf{U}_n^{(1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_n^{(2)}\end{pmatrix},\mathbf{a}_n=\begin{pmatrix}\mathbf{a}_n^{(1)}\\ \mathbf{a}_n^{(2)}\end{pmatrix},$$

$$\mathbf{K}=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}\mathbf{A}^{(1)} & \mathbf{A}^{(2)}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\quad \mathbf{f}(\mathbf{u})=\begin{pmatrix}\mathbf{f}^{(1)}\\ \mathbf{f}^{(2)}\left(\mathbf{u}^{(2)};\mathbf{u}^{(1)}\right)\end{pmatrix}. \qquad (26)$$

where $\mathbf{I}$ – identical and $\mathbf{0}$ – zero operators; block length of vectors and matrices can be defined by vectors length $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$.

Let to substitute the approximation (24) into the differential equation (25) and in compliance with the weighted residuals method let to use the orthogonality of equation residual to tested functions [13]:

$$\int_Y \mathbf{V}_n\cdot\left[\mathbf{K}\cdot\left(\mathbf{U}_n\cdot\dot{\mathbf{a}}_n\right)+\mathbf{A}\cdot\left(\mathbf{u}_{\upsilon}+\mathbf{U}_n\cdot\mathbf{a}_n\right)=\mathbf{f}\left(\mathbf{u}_{\upsilon}+\mathbf{U}_n\cdot\mathbf{a}_n\right)\right]dY=0\,. \qquad (27)$$

Here $\mathbf{V}_n=\left(\mathbf{v}_1\ \ \mathbf{v}_2\ \ \ldots\ \ \mathbf{v}_n\right)$ – matrix, composed from tested functions, that are linearly independent elements of adjoint space to space with element $\mathbf{f}(\mathbf{u})$. Expression (27) can be considered as generalized presentation of the different numerical methods, which are different in choice of trial and tested functions. Method of weighted residuals (27) is equivalent to finite element method in case of using of local trial and tested functions. Orthogonality condition (27) corresponds to the Bubnov–Galerkin method [13], if we are using global trial functions and tested functions have the form $\mathbf{V}_n=\mathbf{U}_n^T$. Bubnov–Galerkin method, as in [14], will lead to algebraic and differential equations after substitution matrices (26) into (27), with taking to account of the operators linearity

$$\mathbf{A}_n^{(1)}\cdot\mathbf{a}_n^{(1)}+\mathbf{A}_n^{(2)}\cdot\mathbf{a}_n^{(2)}=\mathbf{f}_n^{(1)}, \qquad (28)$$

$$\mathbf{K}_n \cdot \dot{\mathbf{a}}_n^{(2)} = \mathbf{f}_n^{(2)}\left(\mathbf{a}_n^{(2)}; \mathbf{a}_n^{(1)}\right), \quad \mathbf{a}_n^{(2)}(0) = \mathbf{0}, \tag{29}$$

where matrices and vectors can be calculated as follow:

$$\mathbf{A}_n^{(2)} = \int_Y \left(\mathbf{U}_n^{(1)}\right)^T \cdot \left(\mathbf{A}^{(2)} \cdot \mathbf{U}_n^{(2)}\right) dY, \qquad \mathbf{K}_n = \int_Y \left(\mathbf{U}_n^{(2)}\right)^T \cdot \mathbf{U}_n^{(2)} dY,$$

$$\mathbf{f}_n^{(2)}\left(\mathbf{a}_n^{(2)}; \mathbf{a}_n^{(1)}\right) = \int_Y \left(\mathbf{U}_n^{(2)}\right)^T \cdot \mathbf{f}^{(2)}\left(\mathbf{U}_n^{(2)} \cdot \mathbf{a}_n^{(2)}; \mathbf{u}_\upsilon^{(1)} + \mathbf{U}_n^{(1)} \cdot \mathbf{a}_n^{(1)}\right) dY.$$

Problem can be presented in canonical form by elimination of vector $\mathbf{a}_n^{(1)}$

$$\dot{\mathbf{a}}_n^{(2)} = \mathbf{b}_n^{(2)}\left(\mathbf{a}_n^{(2)}\right), \quad \mathbf{a}_n^{(2)}(0) = \mathbf{0}, \tag{30}$$

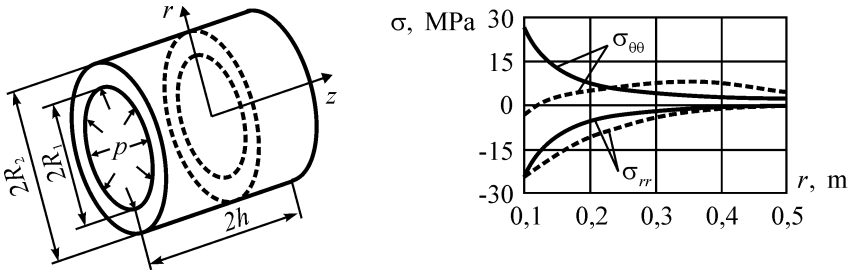
where

$$\mathbf{b}_n^{(2)}\left(\mathbf{a}_n^{(2)}\right) = \mathbf{K}_n^{-1} \cdot \mathbf{f}_n^{(2)}\left(\mathbf{a}_n^{(2)}; \mathbf{a}_n^{(1)}\left(\mathbf{a}_n^{(2)}\right)\right), \ \mathbf{a}_n^{(1)}\left(\mathbf{a}_n^{(2)}\right) = \left(\mathbf{A}_n^{(1)}\right)^{-1} \cdot \left(\mathbf{f}_n^{(1)} - \mathbf{A}_n^{(2)} \cdot \mathbf{a}_n^{(2)}\right).$$

An approach, based on solving differential equations (12), (14) by using the weighted residuals method, leads to approximate solutions. Further, the creep strains and damage parameter can be determined at any point in the body. It is important in time t^* and coordinates x_k^* determination of localized defects formation. That allows to speak about advantage of weighted residuals method to the determination of creep strains and damage parameter in predetermined specially selected points.

3. Examples of numerical solutions of creep theory problems. Example of creep and damage of thick-walled cylinder (fig. 1a) with free edges under the action of internal pressure $p = 25\,\text{MPa}$ is considered as an illustration of stresses redistribution due to creep. Cylinder is made of material D16AT, at $T = 300°\text{C}$. Physical and mechanical constants of the material in creep state equations (10) are taken as: $E = 65\,\text{GPa}$, $B = 0{,}34 \cdot 10^{-7}\ \text{MPa}^{-n}/\text{h}$, $A = 1{,}9 \cdot 10^{-7}\ \text{MPa}^{-m}/\text{h}$, $n = m = 2{,}93$, $\nu = 0{,}3$, $\rho = 3000\ \text{kg}/\text{m}^3$. Geometric parameters of the cylinder in creep solutions are selected to equal: $R_1 = 0{,}1\,\text{m}$, $R_2 = 0{,}5\,\text{m}$ – the inner and outer radiuses respectively, $2h = 0{,}6\,\text{m}$ – cylinder length. Stresses distributions in the cylinder central section $z = 0$ are shown in fig. 1b at initial time moment and before the cylinder rupture. It should be noted a significant redistribution of circumferential stresses due to creep on the inner and outer surfaces.

Numerical solutions are obtained by using Reissners mixed functional (6), (7) and integration of ordinary differential equations with initial conditions (20) by Runge–Kutta–Merson method [9, 15]. Results reliability is confirmed by comparison of calculated results for steady-state creep data of cylinder with well known analytical solutions in the literature [16].



a b

Fig. 1. Cylinder (a) and distributions (b) of radial (σ_{rr}) and circumferential ($\sigma_{\theta\theta}$) stresses for initial time $t = 0$ (solid) and rupture time $t = t^*$ (dash)

Let us consider some mechanisms for the formation time and macroscopic defects location during creep process on example of rectangular plate's tension with parabolic loading in the plane $p = S\left(1 - x_2^2 / b^2\right)$, $S = 240$ MPa , $b = 5$ mm (Fig. 2). Plate material – chromium–nickel stainless steel of 18–8 type [11], with taking to account damage parameter during creep process at temperature $T = 500°\text{C}$. Numerical solution consist from integration of the Cauchy problem (30) for approximations coefficients of stresses, displacements, creep strains and damage parameter, i.e. for the equations. Solutions reliability is based on coincidence with a known analytical solution for elastic deformation of the plate [17] and solutions convergence with increasing number of trial functions.

Calculations results of damage parameter field of the plates (Fig. 2) show the formation of macroscopic defects 2 at time $t = t^*$ in the plate with coordinates $x_1 = \pm a_*$, $x_2 = 0$. Field of damage parameter have been shown on Fig. 2 and based on the grid in plate plane with 103×103 nodes. Integration of damage parameter in predetermined nodes by using (20) lead to the Cauchy problem with much more numbers of equations, than the integration coefficients of approximations by using scheme (30).

Results for damage parameter of plates with different ratio a/b of length and width are shown in Fig. 3. It can be shown that time and coordinate of macroscopic defects formation are limited with increasing of plate length.

4. Conclusions. According to research results some conclusions can be done. For solving of creep problems are more suitable the state space with displacements and stresses, since they lead to the boundary conditions in more simple form without creep strains. By using operator form the mathematical formulations of creep problems can be presented uniformly, which does not depend from state space choice. Numerical solution of initial-boundary-value creep problems by using the known numerical methods – Ritz method or more common method of weighted residuals can be reduced to the integration of

ordinary differential equations with initial conditions. Creep strains and damage parameter can be

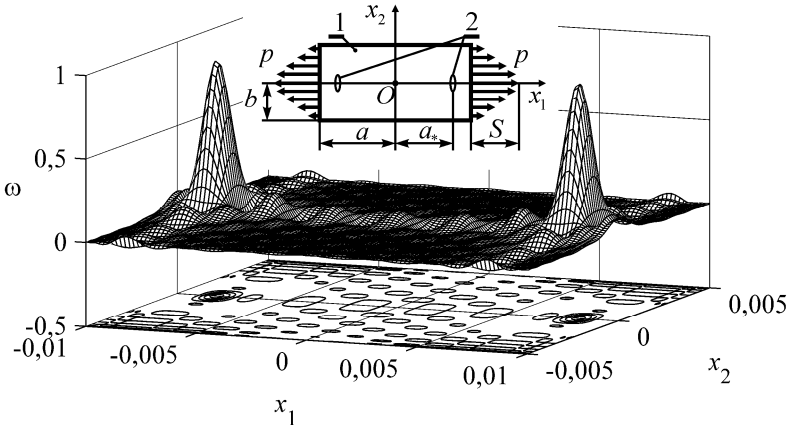


Fig. 2. Plate (1) with defect formation (2) and field of damage parameter in plates points with $a/b = 2$ at the time moment $t = t^*$

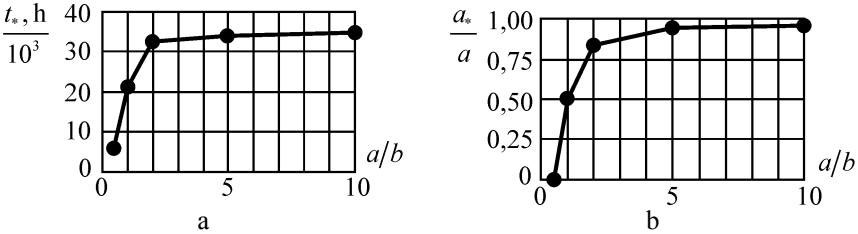


Fig. 3. Dependences of time formation (a) and coordinate formation (b) of defect from plate aspect ratio

determined in numerical solution directly to individual predetermined points, or indirectly through the approximations coefficients. Damage parameter can be calculated at any point by using approximations. It is important in coordinates determining of localized defect. Given examples of numerical solution of creep problems are illustrated effects of significant redistribution for stresses due to creep process and the body size influence at the time and place of macroscopic defects formation.

REFERENCES

1. Superalloys II: High-temperature materials for aerospace and industrial power installations / Ed. Sims Ch.T., Stoloff N.S., Hagel W.K.: Transl. from English. In 2 books. Book. 1 / Ed. Shalin R.E. – Moscow: Metallurgiya, 1995. – 384 p. (in Russian)

2. Lemaitre J., Desmorat R. Engineering Damage Mechanics: Ductile, Creep and Brittle Failures. – Springer–Verlag Berlin Heidelberg, 2005. – 380 p.
3. Bazhenov V.A., Gulyar A.I., Maiboroda E.E., Piskunov S.O. Semianalytic Finite Element-Method in Continuum Creep Fracture Mechanics Problems for Complex-Shaped Spatial Bodies and Related Systems. Part 1. Resolving Relationships of the Semianalytic Finite-Element Method and Algorithms for Solving the Continuum Creep Fracture Problems // Strength of Materials. – 2002. – v. 34, N 5. – P. 425–433.
4. Sklepus A. N., Sklepus N. G. Investigation of the Creep of Plates of Complex Shape by the Method of R–Functions // Strength of Materials. – 2005. – v. 37, N 3. – P. 304–309.
5. Khoroshun L.P. Thermodynamic and statistical foundations of theory of elastoviscoplastic deformation and hardening materials // Math. Methods and Phys.–Mech. Fields. – 2008. – v. 51, N 3. – P. 52–68. (in Russian)
6. Kalyutik A.I., Mosin E.F., Sokovishin Y.A., Entin E.P. Computer calculations of temperature fields in solids with movable boundaries. – L.: Leningrad Univ. Press, 1987. – 224 p. (in Russian)
7. Gorshkov A.G., Rabinsky L.N., Tarlakovsky D.V. Fundamentals of tensor analysis and continuum mechanics: Textbook for high schools. – Moscow: Nauka, 2000. – 214 p. (in Russian)
8. Abovskiy N.P., Andreev N.P., Deruga A.P. The variational principles of elasticity and shell theory. – Moscow: Nauka, 1978. – 288 p. (in Russian)
9. Morachkovsky O.K., Sobol V.N. Problem statements of dynamic creep for isotropic and orthotropic bodies // Materials of the 3rd International Conference "NONLINEAR DYNAMICS" dedicated to the 125th anniversary of the NTU "KhPI", Kharkov. – 2010. – P. 487–491.
10. Pisarenko G.S., Mozharovsky N.S. Equations and boundary-value problems of plasticity and creep theory: Handbook. – K.: Nauk. Dumka, 1981. – 496 p. (in Russian)
11. Morachkovskyii O.K., Romashov Yu.V. Prediction of the Corrosion Cracking Structures Under the Conditions of High–Temperature Creep // Material Science. – 2011. – v. 46, N 5. – P. 613–618.
12. Rvachev V.L. The R–functions theory and some it's applications. – Kiev: Naukova Dumka, 1982. – 522 p. (in Russian)
13. Fletcher K. Numerical methods based on the Galerkin method: Trans. from Eng. – Moscow: Mir, 1988. – 352 p. (in Russian)
14. Morachkovskyii O.K., Romashov Yu.V. Solving Initial-Boundary-Value Creep Problems // Intern. Appl. Mech. – 2009. – v. 45, N 10. – P. 1061–1070.
15. Lance G.N. Numerical Methods for High Speed Computers. – London: Iliffe & Sons Ltd, 1960. – 166 p.
16. Malinin N.N. Creep solutions of machine building constructions elements. – Moscow: Mashinostroenie, 1981. – 221 p. (in Russian)
17. Meleshko V.V. Equilibrium of Elastic Rectangle: Mathieu–Inglis–Pickett Solution Revisited // J. Elasticity. – 1995. – v. 40. – P. 207–238.