EXPONENT MATRICES AND THEIR QUIVERS

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A notion of an exponent matrix is arisen from Ring theory. Exponent matrices are used in applied tasks, for example, by planning multifactor experiments under given optimum conditions, for a construction of codes et cetera. Each exponent matrix can be associated with some graph called a quiver, and we can study such matrices by methods of Graph theory. A notion of a quiver was introduced by P. Gabriel in an investigation of finite–dimensional algebras with the square radical equals zero. A quiver of a reduced exponent matrix is simple laced (without multiple arrows and multiple loops) strongly connected oriented graph. It is also known that there are examples of such graphs that can't be a quiver of exponent matrices [1]. The aim of this paper is to describe graphs that are quivers of exponent matrices. We introduce also a concept of a recursive path at a quiver, and we investigate a relation between a recursive path at the quiver of an exponent matrix and the triangularity of this exponent matrix.

The widespread use of methods of Computational Algebra is an important feature of this work. The named author elaborated computational programs, and some two thousand examples of exponent matrices and their quivers are constructed with these programs. Hypotheses on properties of exponent matrices and their quivers were based on an analysis of these examples. Obtained results were tested with these examples too.

Let $M_n(Z)$ be a ring of square $n \times n$ –matrices over the ring of integers.

A matrix $\varepsilon = (\alpha_{ij})$ from the ring $M_n(Z)$ is called *an exponent matrix* if the following conditions hold:

(i) $\alpha_{ii} = 0$ for all $i = 1, 2, \ldots, n$;

(ii) $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ for all $i, j, k = 1, 2, \ldots, n$.

Relations (ii) are called ring inequalities.

An exponent matrix $\varepsilon = (\alpha_{ij})$ is called *reduced* if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i \neq j$. Notice that it is enough to consider only reduced exponent matrices.

Let $\varepsilon = (\alpha_{ij})$ be a reduced exponent matrix, E be the identity one. Denote $\varepsilon^{(1)} = \varepsilon + E = (\beta_{ij})$, $\varepsilon^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min_k \left\{ \beta_{ik} + \beta_{kj} - \beta_{ij} \right\}$.

A graph Q is *a quiver* of a reduced exponent matrix ε if the adjacency matrix of Q is equal to $\varepsilon^{(2)} - \varepsilon^{(1)}$.

Two exponent matrices are called *equivalent* if one of them can be obtain from other one under transformations of following two forms:

1) subtraction an integer from each element of some line and addition at the same time this integer to each element of the column with the same number;
2) permutation two lines and two columns with the same number simultaneously.

Note that each exponent matrix is equivalent to some exponent matrix with nonnegative elements. Note also that by first transformations a quiver of an exponent matrix does not change, and by second transformations numbering of vertices change but the form of a quiver does not change.

In this paper we consider a problem of describing graphs being quivers of reduced exponent matrices.

Let G be a simple laced strongly connected oriented graph on n $(n \geq 2)$ vertices. We have to find conditions, under which graph G is (is not) a quiver of some reduced exponent matrix.

Theorem 1. *Let* G *be a simple laced strongly connected oriented graph on* n $(n \geq 2)$ *vertices. If* G *has a loop at each vertex, then* G *is a quiver of some reduced exponent matrix.*

Proof. Construct an exponent matrix $\varepsilon = (\alpha_{ij})$ in such way: α_{ij} $(i \neq j)$ is equal to the length of the shortest path from a vertex i to a vertex j of the graph G; $\alpha_{ii} = 0$. We shall prove that the graph G is a quiver of this exponent matrix.

Suppose $\varepsilon^{(1)} = (\beta_{ij})$, $\varepsilon^{(2)} = (\gamma_{ij})$, $[Q(\varepsilon)] = (q_{ij})$. Since the length of the shortest path from one vertex of the quiver G to other one is not less than 1, then for elements of the exponent matrix that do not stand at the main diagonal we have $\alpha_{ij} \geq 1$ for all $i \neq j$, for all elements β_{ij} of the matrix $\varepsilon^{(1)}$ we have $\beta_{ij} \geq 1$, for all elements γ_{ij} of the matrix $\varepsilon^{(2)}$ we have $\gamma_{ij} \geq 2$. Then $\gamma_{ii} = 2$ for all i, since always $\gamma_{ii} \geq 2$.

Consider an element $q_{ss} = \gamma_{ss} - \beta_{ss}$ for any s. Since $\beta_{ss} = 1$ and $\gamma_{ss} = 2$, we have $q_{ss} = 1$. It means that the quiver $Q(\varepsilon)$ has a loop at each vertex.

Suppose graph G has an arrow from a vertex u to a vertex v $(u \neq v)$. It means that the length of the shortest path from the vertex u to the vertex v of the graph G equals to 1, consequently $\alpha_{uv} = \beta_{uv} = 1$. Then $\gamma_{uv} \leq \beta_{uv} + \beta_{vv} = 2$, from where, taking into account the inequality $\gamma_{ij} \geq 2$, we have $\gamma_{uv} = 2$. Then $q_{uv} = \gamma_{uv} - \beta_{uv} = 1$. It means that the quiver $Q(\varepsilon)$ has an arrow from the vertex u to the vertex v.

Thus, we proved the inclusion $G \subseteq Q(\varepsilon)$.

We shall prove the inverse inclusion. We assume that the quiver $Q(\varepsilon)$ has an arrow from a vertex u to a vertex v $(u \neq v)$, i.e. $q_{uv} = 1$ or

$q_{uv} = \min_k \{\beta_{uk} + \beta_{kv} - \beta_{uv}\} = 1$, from where $\beta_{uk} + \beta_{kv} - \beta_{uv} \geq 1$ for all k. Then for each $k \notin \{u; v\}$ we obtain $\alpha_{uk} + \alpha_{kv} - \alpha_{uv} \geq 1$ or $\alpha_{uk} + \alpha_{kv} > \alpha_{uv}$.

Suppose $\alpha_{uv} \geq 2$, i.e. from the vertex u to the vertex v of the graph G there is no arrow, and the shortest path between these vertices has a length not less than 2. Denote by k_0 a vertex of this path first after the vertex u. Obviously, that $k_0 \neq v$, and length of a path from the vertex k_0 to the vertex v equals $\alpha_{k_0 v}$. Consequently, there is $k_0 \notin \{u; v\}$ such that $\alpha_{uk_0} + \alpha_{k_0 v} = \alpha_{uv}$. That contraries to the inequalities $\alpha_{uk} + \alpha_{kv} > \alpha_{uv}$ for all $k \notin \{u; v\}$. Thus $\alpha_{uv} < 2$. Then, taking into account the inequality $\alpha_{uv} \geq 1$, we obtain $\alpha_{uv} = 1$. It means that the graph G has an arrow from the vertex u to the vertex v $(u \neq v)$.

Thus, we proved the inclusion $Q(\varepsilon) \subseteq G$. From the proved inclusions we have $Q(\varepsilon) = G$. Theorem is proved.

Theorem 2. *All possible quivers of reduced exponent matrices on* n *vertices with* n *loops can be obtained from a consideration of exponent matrices, all elements of that are less than or equal* $n-1$.

Proof of Theorem 2 follows from the proof of Theorem 1. Indeed all elements of exponent matrix ε constructed in proving Theorem 1 are less than or equal $n-1$.

Theorem 3. *A simple laced strongly connected oriented graph on* n $(n \geq 2)$ *vertices that has a loop at each vertex except for the one can't be a quiver of a reduced exponent matrix.*

Proof. Assume the converse. Then there is an exponent matrix ε such that its quiver $Q(\varepsilon)$ on n vertices has $n-1$ loops. Without loss of generality we assume that the quiver $Q(\varepsilon)$ has a loop at each vertex except for n. Let $\varepsilon^{(1)} = (\beta_{ij})$, $[Q(\varepsilon)] = (q_{ij})$. Then $q_{nn} = \min_k \{\beta_{nk} + \beta_{kn} - \beta_{nn}\} = 0$, from where it follows that there is k_0 $(k_0 \neq n)$ such that $\beta_{nk_0} + \beta_{k_0 n} = 1$. Taking into account equalities $\beta_{nn} = \beta_{k_0 k_0} = 1$, we obtain $q_{k_0 k_0} = \min_k \{\beta_{k_0 k} + \beta_{kk_0} - \beta_{k_0 k_0}\} = 0$, i.e. the quiver $Q(\varepsilon)$ does not have a loop at the vertex k_0. This contradicts the assumption that the quiver $Q(\varepsilon)$ has loop at each vertex except for n. Theorem is proved.

Suppose n and m are any integer that satisfy the conditions $n \geq 3$, $0 \leq m \leq n-2$. Then there are oriented graphs on n vertices with m loops that are quivers of tiled orders, and there are strongly connected oriented graphs on n vertices with m loops, without multiple loops and multiple arrows that can not be quivers of tiled orders. In [2, 3], we give methods, which enable us to obtain such graphs and corresponding exponent matrices.

We give an example of an exponent matrix with quiver on n $(n \geq 2)$ vertices that has m $(m \leq n-2)$ loops:

$$\varepsilon = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right),$$

where A_{11} and A_{22} are square matrices of order m and $n-m$ respectively :

$$A_{11} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix}; \quad A_{22} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

$A_{12} = (1)$, and $A_{21} = (1)$ are matrices of order $m \times (n-m)$ and $(n-m) \times m$ respectively with elements equal one. If $m = 0$, then matrices A_{11}, A_{12}, A_{21} are missing.

An adjacent matrix of the quiver $Q(\varepsilon)$ is $n \times n$ –matrix:

$$\left[Q(\varepsilon_m)\right] = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right),$$

where

$$C_{11} = \begin{pmatrix} 1 & \dots & 1 & 1 \\ 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & 1 & 1 \end{pmatrix}; \quad C_{12} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 \end{pmatrix};$$

$$C_{21} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}; \quad C_{22} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \dots & \ddots & \ddots & \dots \\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

From a form of the matrix $\left[Q(\varepsilon_m)\right]$ it follows that the quiver $Q(\varepsilon_m)$ on n $(n \geq 2)$ vertices has m $(m \leq n-2)$ loops.

Next Theorem can give strongly connected oriented graphs on n vertices, with m $(m \leq n-2)$ loops, without multiple arrows and multiple loops that can't be quivers of exponent matrices.

Theorem 4. *Suppose* ε *is an exponent matrix, its quiver* $Q(\varepsilon)$ *on* n *vertices has* m *loops* $(n \geq 3; 0 \leq m \leq n)$, K_1 *is the number of arrows (without loops) of the quiver* $Q(\varepsilon)$. *If* $K_1 > n^2 - 2n + 2$, *then each vertex has a loop, i.e.* $m = n$.

Will consider triangular reduced exponent matrices with nonnegative elements and their quivers [4]. Since a quiver of an exponent matrix is a simple laced graph, we will denote a path at a quiver by a following form:

$$v_1 \to v_2 \to v_3 \to \ldots \to v_s,$$

where $v_1, v_2, \ldots, v_s$ are vertices of the quiver.

If all vertices are different, a path is a simple path of the length s denoted by L_s, and if $v_1 = v_s$ and all other vertices are different, a path is a simple cycle denoted by C_{s-1}.

Now we introduce the following concept.

Definition. A simple path L_n of any quiver Q on n vertices will be called *a recursive path* if the following conditions hold:

(i) all vertices of the quiver Q belong to path L_n;

(ii) the quiver Q has no arrows from a vertex of the path L_n to all following vertices of L_n except for next one.

Note that recursive path of a quiver exist if the number of vertices of a quiver is not less than 3, and a quiver can have more than one recursive path. For example, a simple cycle C_n has n recursive paths, a graph G with an adjacent matrix

$$[G] = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix}$$

has two recursive paths.

Theorem 5. *An exponent matrix* ε *is triangular up to an equivalence if and only if its quiver* $Q(\varepsilon)$ *has a recursive path.*

It means that an adjacent matrix of the quiver $Q(\varepsilon)$ up to transposition transformations has a form:

$$[Q(\Lambda)] = \begin{pmatrix} * & 1 & 0 & & 0 \\ & * & 1 & \ddots & \\ & & * & \ddots & 0 \\ & * & & \ddots & 1 \\ & & & & * \end{pmatrix}, \quad (1)$$

where elements that equal 0 or 1 are denoted by the symbol " $*$ ".

We need the following lemma for proofing of Theorem 5.

Lemma. *Let ε be a reduced exponent matrix, and let the quiver $Q(\varepsilon)$ has a recursive path*

$$L_n : v_1 \to v_2 \to v_3 \to \ldots \to v_s \to v_n .$$

Then at $j > i$ for elements of the exponent matrix ε relations

$$\left.\begin{aligned} \alpha_{ij} &= \alpha_{ik} + \alpha_{kj} \quad \text{for } i < k < j; \\ \alpha_{ij} &< \alpha_{ik} + \alpha_{kj} \quad \text{for } k < i \text{ or } k > j \end{aligned}\right\}. \quad (2)$$

hold true.

Proof. Any element q_{ij} $(i \neq j)$ of an adjacent matrix $[Q(\varepsilon)] = (q_{ij})$ is defined by a formula $q_{ij} = \min\limits_{k:k\neq i,k\neq j}\{1;\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\}$. From this expression it follows that if the quiver $Q(\varepsilon)$ does not have an arrow from a vertex i to a vertex j (i.e. $q_{ij} = 0$), there is always a number k_0 $(k_0 \neq i, k_0 \neq j)$ such that the equality $\alpha_{ij} = \alpha_{ik_0} + \alpha_{k_0 j}$ holds true, and if the quiver $Q(\varepsilon)$ has an arrow from a vertex i to a vertex j (i.e. $q_{ij} = 1$), the inequalities $\alpha_{ij} < \alpha_{ik} + \alpha_{kj}$ hold true for all k $(k_0 \neq i, k_0 \neq j)$.

Since $q_{i,i+1} = 1$, in case $j = i+1$ we have $\alpha_{i,i+1} < \alpha_{ik} + \alpha_{k,i+1}$ for all k: $k \neq i$, $k \neq i+1$, i.e. in case $j = i+1$ relations (2) hold true.

In case $j > i+1$ relations (2) are proved by mathematical induction on pair (i,j). A set of pairs (i,j) is considered as lexicographic ordered one: $(1,3),(1,4),\ldots,(1,n),(2,4),(2,5),...,(2,n),\ldots,(n-2,n)$. Base of induction and inductive step are proved by contradiction.

This completes the proof of Lemma.

Proof of Theorem 5. *Necessity*. Suppose $\varepsilon = (\alpha_{ij})$ is a lower triangular exponent matrix, $Q(\varepsilon)$ is its quiver, and $[Q(\varepsilon)] = (q_{ij})$ is the adjacent matrix of the quiver $Q(\varepsilon)$. We consider an element q_{mj} over the first upper diagonal ($j \geq m+2$): $q_{mj} = \min\limits_{k:k\neq m,k\neq j}\{1;\alpha_{mk} + \alpha_{kj} - \alpha_{mj}\}$. Taking into account the triangularity of the exponent matrix ε, for $k = m+1$ we have $\alpha_{m,m+1} + \alpha_{m+1,j} - \alpha_{mj} = 0$, from where $q_{mj} = 0$, i.e. for all $m = 1, 2, \ldots, n-2$ there are not arrows from a vertex m of the quiver $Q(\varepsilon)$ to vertices with numbers greater than $m+1$.

Consider an element $q_{m,m+1}$ on first upper diagonal of the adjacent matrix $[Q(\varepsilon)]$: $q_{m,m+1} = \min_{k:k\neq m, k\neq m+1}\{1; \alpha_{mk} + \alpha_{k,m+1} - \alpha_{m,m+1}\}$. Taking into account the triangularity and the reduction of the exponent matrix ε, for $k < m$ we have $\alpha_{mk} + \alpha_{k,m+1} - \alpha_{m,m+1} = \alpha_{mk} + 0 - 0 = \alpha_{mk} \geq 1$, and for $k > m+1$ we have $\alpha_{mk} + \alpha_{k,m+1} - \alpha_{m,m+1} = 0 + \alpha_{k,m+1} - 0 = \alpha_{k,m+1} \geq 1$, from where we obtain $q_{m,m+1} = 1$, i.e. there always is an arrow from each vertex m of the quiver $Q(\varepsilon)$ to a vertex $m+1$ for all $m = 1, 2, \ldots, n-1$.

Thus a lower triangular exponent matrix always has a recursive path. A proof for a upper triangular exponent matrix is similar.

Sufficiency. Assume that a quiver $Q(\varepsilon)$ satisfies the condition of the Theorem 5, that is its adjacent matrix has a form (1). Will prove that the exponent matrix ε is equivalent to a triangular one. Will do by proving only followings transformations of the exponent matrix ε: will at the same time subtract an integer from some line and add this integer to the column with the same number. By such transformations of the matrix ε ring inequalities remain inequalities and equalities do equalities. Thus the quiver $Q(\varepsilon)$ does not change.

Consider all possible cases of a form of the exponent matrix $\varepsilon = (\alpha_{ij})$:

(i) $\alpha_{1n} = 0$. Then from Lemma it follows that $\alpha_{ij} = 0 \ \forall i, j : j > i$, that is the exponent matrix ε is triangular.

(ii) $\alpha_{12} = \alpha_{13} = \ldots = \alpha_{1,n-1} = 0 \neq \alpha_{1n} := \mu_1$. Then from Lemma it follows that $\alpha_{ij} = 0$ for all $i < j < n$ and $\alpha_{in} = \mu_1$ for all $i \neq n$. Will subtract the integer μ_1 from the elements of the n-th column, and will add this integer to the elements of the n-th line. We obtain a triangular exponent matrix.

(iii) $\alpha_{12} = \alpha_{13} = \ldots = \alpha_{1,n-1} := \mu_2 \neq 0$; $\alpha_{1n} > \mu_2$. Will subtract the integer μ_2 from the elements of the first line, and will add this integer to the elements of the first column. We obtain an exponent matrix that satisfies the condition (ii).

(iv) $\alpha_{12} = \alpha_{13} = \ldots = \alpha_{1n} := \mu_3 > 0$. Will subtract the integer μ_3 from the elements of the first line, and will add this integer to the elements of the first column. We obtain a triangular exponent matrix.

(v) $\alpha_{12} = \alpha_{13} = \ldots = \alpha_{1p} := \mu_4 < \alpha_{1,p+1}$ $(2 \leq p \leq n-2)$. Will subtract the integer μ_4 from the elements of the first line, and will add this integer to the elements of the first column. Then will apply the following. Will sequentially for $k = p, p-1, \ldots, 2$ subtract an integer $\nu_k := \min\{\alpha_{k,k-1}, \alpha_{k,k+1}\}$ from the elements of a k-th line, and will add this integer to the elements of a k-th column. By ring inequalities, Lemma, and reduction of the exponent matrix ε, we have $\nu_p \geq \nu_{p-1} \geq \ldots \geq \nu_2 > 0$. We obtain a matrix that has $\alpha_{12} =: \mu_5 > 0$. Will subtract the integer μ_5 from the elements of the first line, and will add this integer to the elements of the first column. As a result of such procedure we obtain one of cases (i), (ii) or (v). A value α_{1n} is subtracted on a positive integer. If we obtain the case (v), we will apply procedure given above again.

Note that by specified transformations, elements of exponent matrices are nonnegative. As a result we obtain a triangular exponent matrix after a finite number of transformations.

So the exponent matrix ε is equivalent to a triangular one.

Theorem 5 is proved.

The obtained results are theoretical, and they can be used for an investigation in the representation theory and in the modern structural theory of rings. These results can be also used by giving a specialized course in algebra. In the future we plan to carry out further investigation of the formulated problem.

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