INDEXES OF SEMIPERFECT RINGS

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A notion of an index of a semiperfect ring was introduce to investigate a structure of such rings [1]. In this paper we research indexes of various classes of semiperfect rings [2]. We use methods of Quiver Theory and Ring and Module Theory.

All rings, which are considered in this paper, are associative, $1 \neq 0$, and all A–modules are right and unitary.

Let A be a ring, let $R = \mathrm{Rad}\, A$ be the Jacobson radical of the ring A.

A ring A is called *semiperfect* if the factorring A/R is Artinian and idempotents can be lifted modulo the radical R.

A ring A is right *semidistributive* (left semidistributive) if right regular module A_A (left regular module ${}_A A$) is semidistributive. A right semidistributive and left semidistributive ring is *semidistributive*.

A ring A is *semiprime* if 0 is a semiprime ideal in A, i.e. A does not contain nonzero nilpotent ideals.

A ring A is *prime* if 0 is a prime ideal in A, i.e. the product of any two nonzero ideals in A is not equal to zero.

A ring A is *weakly prime* if the product of any two-sides ideals that are not contained to the Jacobson radical R of the ring A is not equal to zero.

Prime rings of the form

$$\Lambda = \begin{pmatrix} \vartheta & \pi^{\alpha_{12}}\vartheta & \cdots & \pi^{\alpha_{1n}}\vartheta \\ \pi^{\alpha_{21}}\vartheta & \vartheta & \cdots & \pi^{\alpha_{2n}}\vartheta \\ \cdots & \cdots & \cdots & \cdots \\ \pi^{\alpha_{n1}}\vartheta & \pi^{\alpha_{n2}}\vartheta & \cdots & \vartheta \end{pmatrix}, \qquad (1)$$

are called *tiled orders*. Here $n \geq 1$, ϑ is a discrete valuation ring with a prime element π, α_{ij} are integer, $\alpha_{ii} = 0$ for all i, and $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ for all i, j, k (these relations are called ring inequalities).

We can assume all α_{ij} are nonnegative integers up to an isomorphism.

Note that we consider a ring ϑ embedded in its skew field D.

Denote by $M_n(D)$ the ring of all square matrices of order n with coefficients from a skew field D.

A ring Λ of the form (1) with the classical ring of quotients $Q = M_n(D)$ shall be written as $\Lambda = \{\vartheta, \varepsilon(\Lambda)\}$, where $\varepsilon(\Lambda) = (\alpha_{ij})$ is an exponent matrix, ϑ is a ring with a skew field D.

Semiperfect ring A is called *reduced* if the factorring A/R is a direct sum of skew fields. It means that a decomposition of the ring A into direct sum of the principal $A-$ module has not isomorphic modules. Any semiperfect ring is Morita equivalent to some reduced ring. Tiled order $\Lambda=\{\vartheta,\varepsilon(\Lambda)\}$ is reduced if and only if the exponent matrix $\varepsilon(\Lambda)=(\alpha_{ij})$ has not symmetric zeros.

Suppose A is a semiperfect right Noetherian ring; R is its Jacobson radical, P_1, P_2, … , P_n are all pairwise nonisomorphic indecomposable projective modules. Suppose a projective cover $P(P_iR)$ of the module P_iR is

$$P(P_iR)=\overset{n}{\underset{j=1}{\oplus}}P_j^{t_{ij}}\ ,\ \ i,j=1,2,\dots,n\ .$$

We put points (vertices) $1,2,\dots,n$ in correspondence to the modules P_1, P_2, … , P_n, and join a vertex with a vertex by t_{ij} arrows.

The constructed graph is called *a quiver* of a semiperfect right Noetherian ring A denoted $Q(A)$. Note that the quiver of a semiperfect right Noetherian ring does not change by switching to a Morita equivalent ring.

Now we introduce the concept of an index of a semiperfect ring.

Definition. Suppose A is a semiperfect ring, $Q(A)$ is the quiver of the ring A, $[Q(A)]$ is the adjacency matrix of the quiver $Q(A)$. The maximal real eigenvalue of the matrix $[Q(A)]$ will be called *an index* and will be denote by $\operatorname{in}A$.

Theorem 1. *Let* A *be a semiperfect semidistributive ring. Then* $0\le \operatorname{in}A\le n$, *where* n *is the number of vertices of the quiver* $Q(A)$.

Proof of Theorem 1 follows from the Frobenius theorem and from the structure of the quiver of such ring.

Theorem 2. *For any integer* i $(1\le i\le n)$ *there is a semiperfect semidistributive ring* A *such that its quiver* $Q(A)$ *has* n *vertices and* $\operatorname{in}A=i$.

Proof. Consider a case $n\ge 3$. Construct a graph G with a set of vertices $VG=\{1,2,\dots,n\}$ and with a set of arrows AG in such way: an arrow $\sigma: j\to k$ $(j,k=1,2,\dots,n,\ j\ne k)$ belongs to the set AG if either of the two conditions holds:

$$\text{(i)}\quad \begin{cases}1\le j\le n-i,\\ j<k\le i+j;\end{cases}\qquad\qquad \text{(ii)}\quad \begin{cases}n-i<j\le n,\\ \left[\begin{array}{l} j<k\le n,\\ k\le i+j-n.\end{array}\right.\end{cases}$$

Consider the path algebra $A(G)$ of this finite graph G. We denote by J the fundamental ideal of the algebra $A(G)$, i.e. an ideal generated by all arrows of G. Let $A_i=A(G)/J^2$. It is clear that the algebra A_i is semidistributive and $Q(A_i)=G$.

The quiver $Q(A_i)$ has n vertices, exactly i arrows go to each vertex of the quiver $Q(A_i)$, and exactly i arrows go from each vertex of the quiver $Q(A_i)$, therefore, the adjacency matrix $[Q(A_i)]$ has in each line exactly i identities and exactly $n-i$ zeros, i.e. it is multiple to a stochastic matrix. Thus $\operatorname{in} A_i = i$.

Case $n = 2$. Consider tiled orders $\Lambda_1 = \{\vartheta, \varepsilon(\Lambda_1)\}$ and $\Lambda_2 = \{\vartheta, \varepsilon(\Lambda_2)\}$ with exponent matrices respectively

$$\varepsilon(\Lambda_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \varepsilon(\Lambda_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easily verified that $\operatorname{in} A_1 = 1$ and $\operatorname{in} A_2 = 2$.

Case $n = 1$ is obvious. The Theorem is proved.

Theorem 3. *Suppose* A *is a weakly prime semiperfect Noetherian ring; then the following conditions are equivalent*:

a) *the ring* A *is serial*;

b) $\operatorname{in} A \le 1$.

Proof. a) $\Rightarrow$ b). Suppose the ring A is serial; then the quiver $Q(A)$ of the ring A is strongly connected. But a quiver of a serial Noetherian ring is a disconnected union of cycles and chains. That is why the quiver $Q(A)$ is either a cycles or a chain. Therefore, either $\operatorname{in} A = 1$ or $\operatorname{in} A = 0$, i.e. $\operatorname{in} A \le 1$.

b) $\Rightarrow$ a). Suppose $\operatorname{in} A \le 1$. Consider two cases.

Case 1. Suppose $Q(A)$ has one vertex without loops; then $A = M_n(D)$, where D is a skew field. Thus A is a simple Artinian ring, which is a serial ring.

Case 2. The quiver $Q(A)$ is a strongly connected graph with arrows. Then the matrix $[Q(A)]$ has not zero lines and not zero columns. Therefore, in this case $\operatorname{in} A$ is always not less than one, so $\operatorname{in} A = 1$. Thus we can number coordinates of a positive eigenvector $\vec{Z} = \{z_1, z_2, \ldots, z_n\}$ such that the inequalities $z_1 \ge z_2 \ge \ldots \ge z_n$ hold. Assume $z_1 = z_2 = \ldots = z_i$ and $z_i > z_{i+1}$; then $\vec{Z} = \{z_1, \ldots, z_1, z_{i+1}, \ldots, z_n\}$.

Let $i = 1$; then $q_{i1} = 0$ for $i \ge 2$. Therefore,

$$[Q(A)] = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots \\ 0 & * & \ldots & * \end{pmatrix}.$$

Let $q_{1j}=1$ for $j\geq 2$; then $z_1+z_j=z_1$. We obtain the contradiction. Therefore, taking into account the indecomposability of the quiver $[Q(A)]$, we obtain $[Q(A)]=(1)$.

Let $i\geq 2$; then $[Q(A)]$ has the following form:

$$[Q(A)]=\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right),$$

where the upper left block has the order $i\times i$. But the matrix $[Q(A)]$ is permutationally irreducible. We obtain the contradiction. Therefore, $z_1=z_2=\ldots=z_n$. That is why in each line there is exactly one identity.

Taking into account the strongly connectivity of the quiver $Q(A)$, we obtain that the quiver $Q(A)$ is a simple cycle, which is related to the ring $H_n(\vartheta)$ with a following exponent matrix

$$\varepsilon\left(H_n(\vartheta)\right)=\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 0 \end{pmatrix},$$

where ϑ is a discrete valuation ring. But such ring is serial. Theorem is proved.

Theorem 4. *Suppose* A *is a semiprime semiperfect semidistributive right Noetherian ring* A*; then the following conditions are equivalent*:

a) *the ring* A *is hereditary*;

b) $\operatorname{in} A\leq 1$.

Proof. a) $\Rightarrow$ b). A hereditary ring A is Morita equivalent to a finite direct product of skew fields and rings of the form $H_n(\vartheta)$, where ϑ is a discrete valuation ring. The quiver of such ring is a disconnected union of points and some number of simple cycles. Thus $\operatorname{in} A\leq 1$.

b) $\Rightarrow$ a). Take into account properties of the ring A, we obtain if $\operatorname{in} A=0$, then the ring A is semisimple, thus A is simple, therefore, A is hereditary. If A is tiled order and $\operatorname{in} A=1$, then the matrix $[Q(A)]$ has exactly one identity in each line and in each column. As the quiver $Q(A)$ is strongly connected, it is a simple cycle, therefore, A is Morita equivalent to the ring $H_n(\vartheta)$.

Theorem is proved.

Theorem 5. *The index of a hereditary Artinian ring is equal zero.*

Proof. A quiver $Q(A)$ of a hereditary Artinian ring is an acyclic quiver without loops. Thus the adjacent matrix of the quiver $Q(A)$ can be reduced to a triangular matrix of the form:

$$[Q(A)] = \begin{pmatrix} 0 & q_{12} & q_{13} & \cdots & q_{1n} \\ 0 & 0 & q_{23} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where n is the number of vertices of the quiver $Q(A)$, $q_{ij} \geq 0$ for all $i, j = \overline{1,n}$ such that $j > i$.

Characteristic polynomial of the matrix $[Q(A)]$ is $\chi(\lambda) = \lambda^n$. Its roots are $\lambda_{1,2,\ldots,n} = 0$, from where we obtain $\mathrm{in}\, A = 0$.

The Theorem is proved.

The obtained results are theoretical, and they can be used for the development of methods of Quiver Theory in the modern structural Ring Theory. These results can be also used by giving specialized courses in algebra. We plan to use the index of a semiperfect ring for a classification of such rings.

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