SOLUTIONS ASSOCIATED WITH BOTH THE BOUND STATE SPECTRUM AND THE SPECIAL SINGULARITY FUNCTION FOR CONTINUOUS SPECTRUM IN INVERSE SCATTERING METHOD

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It is of significance to look for exact solutions of nonlinear evolution equations in many applications of physics and technology. Various effective approaches have been developed to construct exact wave solutions of completely integrable equations. The inverse scattering method is the most appropriate way of tackling the initial value problem [1,2].

This paper deals with a nonlinear evolution equation

$$W_{XXT} + (1 + W_T)W_X = 0, \tag{1}$$

which arises from the Vakhnenko equation (VE) [3–5]

$$(u_t + uu_x)_x + u = 0 \tag{2}$$

through the transformation [6,7]

$$u(x,t) = U(X,T) = W_X(X,T), \quad x = x_0 + T + W(X,T), \quad t = X. \tag{3}$$

These equations describe high-frequency perturbations in a relaxing medium [5]. Following the papers [8,9], hereafter the equation (1) is referred to as the Vakhnenko–Parkes equation (VPE). Hone and Wang [10] have shown that there is a subtle connection between the Sawada–Kotera hierarchy and the VE, between the Degasperis–Procesi equation and the VE.

Recently the inverse scattering method has been applied to obtain the exact *N*–soliton solutions of the VPE [11]. In this paper we use the inverse scattering transform method to study additionally the periodic solutions of the VPE (1) associated with continuum part of the spectral data as well as to investigate the interaction of solitons with these periodic waves.

1. The associated eigenvalue problem for the VPE. In order to use the inverse scattering method, one first has to formulate the associated eigenvalue problem. In [11] it is shown that the pair equations

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{4}$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \tag{5}$$

is associated with the VPE (1). The inverse problem for third-order spectral equations (4) has been considered by Caudrey [12] and Kaup [13]. We adapt the results obtained by these authors to the present spectral problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE. The solution of the linear equation (4) has been found by Caudrey

[12] in terms of Jost functions $\varphi_j(X,\zeta)$ through $\Phi_j(X,\zeta) = \exp\{-\lambda_j(\zeta)X\}\,\varphi_j(X,\zeta)$, $\lambda_j(\zeta)=\omega_j\zeta$, $\lambda_j^3(\zeta)=\lambda$, $\omega_j = e^{2\pi i(j-1)/3}$. The equation (5) determinates T–evolution of the scattering data. It turns out [12,13] that we need only consider the element $\varphi_1(X,\zeta)$ (as well as $\Phi_1(X,\zeta)$). In general case it is necessary to take into account both the bound state spectrum and the continuous spectrum. According to the relation (6.20) in [12], the solution of (4) is as follows

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)})-\lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)})-\lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)}) + \frac{1}{2\pi i}\int\sum_{j=2}^{3}Q_{1j}(\zeta')\frac{\exp\{[\lambda_j(\zeta')-\lambda_1(\zeta')]X\}}{\zeta'-\zeta}\Phi_1^{\pm}(X,\omega_j\zeta')d\zeta'. \quad (6)$$

Eq. (6) contains the spectral data, namely, K poles with the quantities $\gamma_{1j}^{(k)}$ for the bound state spectrum as well as the functions $Q_{1j}(\zeta')$ given along all the boundaries of regular regions for the continuous spectrum. The boundaries between regions, where the Jost function $\varphi_1(X,\zeta)$ is regular, appear at $\mathrm{Re}(\lambda_1(\zeta')-\lambda_j(\zeta'))=0$ over all $j\neq 1$ [12]. The integral in (6) is along all the boundaries (see the dashed lines in Fig. 1).

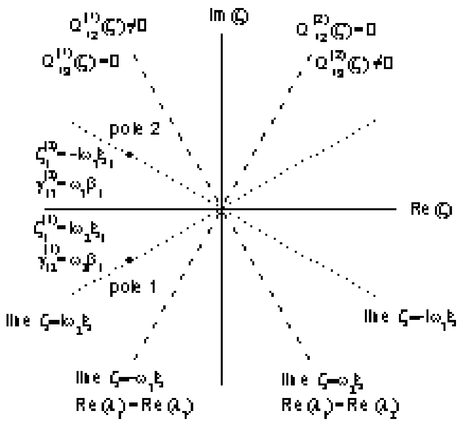

Fig. 1. The regular regions for Jost functions $\varphi_1(X,\zeta)$ in the complex ζ–plane. The dashed lines determine the boundaries between regular regions. These lines are lines where the singularity functions $Q_{1j}(\zeta')$ are given. The dotted lines are the lines where the poles appear.

Provided $Q_{1j}(\zeta)\equiv 0$ in (6), the consideration of the bound state spectrum only gives rises to the purely soliton solutions. The procedure for finding the exact N–soliton solution of the VPE via the inverse scattering method is described in paper [11].

2. Special singularity function for continuous spectrum. Additionally to the bound state spectrum we consider the continuous spectrum of the associated eigenvalue problem, i.e. assume that at least some of the functions $Q_{1j}(\zeta')$ are nonzero. This singularity can appear only on boundaries between the regular regions on the ζ–plane. The condition $\mathrm{Re}(\lambda_1(\zeta')-\lambda_j(\zeta'))=0$ determines these boundaries [12]. According to [12] we find that for $\Phi_1(X,\zeta)$ the complex ζ–plane is divided into four regions by two lines

$$\text{(i) } \zeta'=\omega_2\xi\text{, with } Q_{12}^{(1)}(\zeta')\neq 0,\quad Q_{13}^{(1)}(\zeta')\equiv 0, \tag{7a}$$

$$\text{(ii) } \zeta'=-\omega_3\xi\text{, with } Q_{12}^{(2)}(\zeta')\equiv 0,\quad Q_{13}^{(2)}(\zeta')\neq 0 \tag{7b}$$

where ξ is real (see Fig. 1). Analysis shows that the direction of the integration in (6) is to be so that ξ sweeps from $-\infty$ to $+\infty$.

Let us consider the singularity functions $Q_{1j}(\zeta')$ on the boundaries, on which the Jost function $\varphi_1(X,\zeta)$ is singular, in the form ($n=1,2,...,N$)

$$\left.\begin{aligned} Q_{12}^{(1)}(\zeta') &= -2\pi i\sum_{n=1}^{N} q_{12}^{(2n-1)}\delta(\zeta'-\zeta'_{2n-1}) \\ Q_{13}^{(1)}(\zeta') &= -2\pi i\sum_{n=1}^{N} q_{13}^{(2n-1)}\delta(\zeta'-\zeta'_{2n-1})\equiv 0 \end{aligned}\right\} \text{ on the line } \zeta'=\omega_2\xi, \tag{8a}$$

$$\left.\begin{aligned} Q_{12}^{(2)}(\zeta') &= -2\pi i\sum_{n=1}^{N} q_{12}^{(2n)}\delta(\zeta'-\zeta'_{2n})\equiv 0 \\ Q_{13}^{(2)}(\zeta') &= -2\pi i\sum_{n=1}^{N} q_{13}^{(2n)}\delta(\zeta'-\zeta'_{2n}) \end{aligned}\right\} \text{ on the line } \zeta'=-\omega_3\xi. \tag{8b}$$

For the singularity functions (8) and for M pairs of poles, the relationship (6) is reduced to the form

$$\begin{aligned} \Phi_1(X,\zeta) = 1 &- \sum_{k=1}^{2M}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)})-\lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)})-\lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)}) \\ &- \sum_{l=1}^{2N}\sum_{j=2}^{3} q_{1j}^{(l)}\frac{\exp\{[\lambda_j(\zeta'_l)-\lambda_1(\zeta'_l)]X\}}{\zeta'_l-\zeta}\Phi_1(X,\omega_j\zeta'_l). \end{aligned} \tag{9}$$

In [11] it is proved that the poles appear in pairs only $\zeta_1^{(2m-1)}=i\omega_2\xi_1$, $\zeta_1^{(2m)}=-i\omega_3\xi_1$, under the conditions $\gamma_{12}^{(2m-1)}=\omega_2\beta_m$, $\gamma_{13}^{(2m-1)}=0$, $\gamma_{12}^{(2m)}=0$, $\gamma_{13}^{(2m)}=\omega_3\beta_m$, ($m=1,2,...,M$). Moreover, the singularities in the form (8) appear also in pairs $\zeta'_{2n-1}=\omega_2\xi_n$, $\zeta'_{2n}=-\omega_3\xi_n$ with $q_{12}^{(2n-1)}\omega_2=q_{13}^{(2n)}$ for $n=1,2,...,N$ [14].

Insofar as we have 2M poles and 2N coefficients $q_{12}^{(2n-1)}$, $q_{13}^{(2n)}$ in the adopted specifications (8) of the singularity functions $Q_{1j}(\zeta')$, it is convenient to introduce the notations

$$\mu_{ji}=\begin{cases}\lambda_j(\zeta_1^{(i)})\\ \lambda_j(\zeta'_{(i-K)})\end{cases},\quad p_{1j}^{(i)}=\begin{cases}\gamma_{1j}^{(i)} & \text{at}\quad i=1,...,K\\ q_{1j}^{(i-K)} & \text{at}\quad i=K+1,...,K+L\end{cases}, \tag{10}$$

where $K=2M$ and $L=2N$. Then the relationship (6) are rewritten as follows

$$\Phi_1(X,\zeta)=1-\sum_{i=1}^{K+L}\sum_{j=2}^{3}p_{1j}^{(i)}\frac{\exp[(\mu_{ji}-\mu_{1i})X]}{\mu_{1i}-\zeta}\Phi_1(X,\mu_{ji}). \tag{11}$$

According to [11] the solution of Eq. (1) can be found (see also Eq. (6.38) in [12])

$$W(X)-W(-\infty)=3\frac{\partial}{\partial X}\ln(\det M(X)) \tag{12}$$

through the matrix $M(X)$, which is defined as follows

$$M_{il}(X)=\delta_{il}-\sum_{j=2}^{3}p_{1j}^{(i)}\frac{\exp[(\mu_{ji}-\mu_{1l})X]}{\mu_{ji}-\mu_{1l}}. \tag{13}$$

Now let us consider the T–evolution of the spectral data. By analyzing the solution of Eq. (5) when $X\to-\infty$, we find that $\varphi_j(X,T,\zeta)=\exp[-(3\lambda_j(\zeta))^{-1}T]\varphi_j(X,0,\zeta)$. Hence, the T–evolution of the scattering data is given by the relationships (with $i=1,2,...,K+L$)

$$\lambda_j(T)=\lambda_j(0),\quad p_{1j}^{(i)}(T)=p_{1j}^{(i)}(0)\exp\{[-(3\mu_{ji})^{-1}+(3\mu_{1i})^{-1}]T\}. \tag{14}$$

Consequently, the final result for the solution of the VPE, when we consider the spectral data from both the bound state spectrum and the continuous spectrum, as well as taking into account their T–evolution, is as follows:

$$U(X,T)=W_X(X,T)=3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right). \tag{15}$$

Here $M(X,T)$ is the $(K+L)\times(K+L)$ matrix given by

$$M_{kl}=\delta_{kl}-\sum_{j=2}^{3}p_{1j}^{(k)}\frac{\exp\{(\mu_{jk}-\mu_{1l})X+[-(3\mu_{jk})^{-1}+(3\mu_{1k})^{-1}]T\}}{\mu_{jk}-\mu_{1l}}, \tag{16}$$

where for $i\le M$

$$\begin{aligned}
&\mu_{1(2i-1)}=\lambda_1(\zeta_1^{(2i-1)})=i\omega_2\xi_i, && \mu_{2(2i-1)}=\lambda_2(\zeta_1^{(2i-1)})=i\omega_3\xi_i,\\
&p_{12}^{(2i-1)}=\gamma_{12}^{(2i-1)}=\omega_2\beta_i, && p_{13}^{(2i-1)}=\gamma_{13}^{(2i-1)}=0,\\
&\mu_{1(2i)}=\lambda_1(\zeta_1^{(2i)})=-i\omega_3\xi_i, && \mu_{3(2i)}=\lambda_3(\zeta_1^{(2i)})=-i\omega_2\xi_i,\\
&p_{12}^{(2i)}=\gamma_{12}^{(2i)}=0, && p_{13}^{(2i)}=\gamma_{13}^{(2i)}=\omega_3\beta_i,
\end{aligned} \tag{17}$$

and for $M<i\le M+N$

$$\begin{aligned}
&\mu_{1(2i-1)} = \lambda_1(\zeta'_{2(i-M)-1}) = \omega_2\xi_i, && \mu_{2(2i-1)} = \lambda_2(\zeta'_{2(i-M)-1}) = \omega_3\xi_i, \\
&p_{12}^{(2i-1)} = q_{12}^{(2(i-M)-1)} = \omega_2\beta_i, && p_{13}^{(2i-1)} = q_{13}^{(2(i-M)-1)} = 0, \\
&\mu_{1(2i)} = \lambda_1(\zeta'_{2(i-M)}) = -\omega_3\xi_i, && \mu_{3(2i)} = \lambda_3(\zeta'_{2(i-M)}) = -\omega_2\xi_i, \\
&p_{12}^{(2i)} = q_{12}^{(2(i-M))} = 0, && p_{13}^{(2i)} = q_{13}^{(2(i-M))} = \omega_3\beta_i.
\end{aligned} \tag{18}$$

For the solution (15), (16) there are $(M+N)$ arbitrary constants ξ_i and $(M+N)$ arbitrary constants β_i. The constants ξ_i are real, while the constants β_i, in general case, are complex.

As will be clear from the examples in next section, the solution (15), (16) includes N discrete frequencies from continuum part of the spectral data. For this reason, the solution (15), (16), without solitons (i.e. with $M=0$), will be referred to as N–mode solution of the VPE. Evidently these discrete modes emanate from the special choice (8) of the singularity functions $Q_{1j}(\zeta')$.

3. The soliton and periodic solutions. To obtain the solutions of the VPE, one has to calculate the determinant of matrix (16). We present three results of such calculation for $M+N \le 3$. For the sake of convenience we will use the auxiliary function $F(X,T)$ given by the definition $F(X,T) = \sqrt{\det M(X,T)}$. In particular, from (16),

1) for $M+N=1$ we have

$$F = 1 + c_1 q_1; \tag{19}$$

2) for $M+N=2$ we have

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2; \tag{20}$$

3) for $M+N=3$ we have

$$\begin{aligned}
F = 1 &+ c_1 q_1 + c_2 q_2 + c_3 q_3 + b_{12} c_1 c_2 q_1 q_2 + b_{13} c_1 c_3 q_1 q_3 \\
&+ b_{23} c_2 c_3 q_2 q_3 + b_{12} b_{13} b_{23} c_1 c_2 c_3 q_1 q_2 q_3.
\end{aligned} \tag{21}$$

For $M+N>3$, the explicit expression for the function $F(X,T)$ can be obtained in a similar manner. It is reasonable to present the quantities c_i, q_i, b_{ij} involved in the above formulas (19)–(21) separately for three distinct cases:

(i) the purely solitonic case $(i,j) \le M$ assumes

$$\begin{aligned}
&q_i = \exp(2\theta_i), \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \\
&c_i = \frac{\beta_i}{2\sqrt{3}\xi_i}, \quad b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad b_{ij} \ge 0;
\end{aligned} \tag{22}$$

(ii) the case of purely multi-mode waves $M < (i,j) \le M+N$ assumes

$$q_i = \exp(2\theta_i), \quad 2\theta_i = -i\sqrt{3}\xi_i X + (i\sqrt{3}\xi_i)^{-1} T,$$

$$c_i = \frac{i\beta_i}{2\sqrt{3}\xi_i}, \quad b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad b_{ij} \geq 0; \tag{23}$$

(iii) the case of a combination of solitons $(i, i') \leq M$ and multi-mode waves $M < (j, j') \leq M + N$ ·assumes

$$q_i = \exp(2\theta_i), \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i},$$

$$q_j = \exp(2\theta_j) \quad 2\theta_j = -i\sqrt{3}\xi_j X + (i\sqrt{3}\xi_j)^{-1} T, \quad c_j = \frac{i\beta_j}{2\sqrt{3}\xi_j},$$

$$b_{ii'} = \left(\frac{\xi_i - \xi_{i'}}{\xi_i + \xi_{i'}}\right)^2 \frac{\xi_i^2 + \xi_{i'}^2 - \xi_i\xi_{i'}}{\xi_i^2 + \xi_{i'}^2 + \xi_i\xi_{i'}}, \quad 0 \leq b_{ii'} \leq 1, \tag{24}$$

$$b_{jj'} = \left(\frac{\xi_j - \xi_{j'}}{\xi_j + \xi_{j'}}\right)^2 \frac{\xi_j^2 + \xi_{j'}^2 - \xi_j\xi_{j'}}{\xi_j^2 + \xi_{j'}^2 + \xi_j\xi_{j'}}, \quad 0 \leq b_{jj'} \leq 1,$$

$$b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad |b_{ij}| = 1.$$

With the above found representation of the auxiliary function $F(X,T)$ and taking into account the key relationship (12), we can write the explicit solution to the basic nonlinear evolution equation (1) in the following concise form:

$$W(X,T) = 6\frac{\partial}{\partial X}\ln(F(X,T)) + \text{const}. \tag{25}$$

The function $F(X,T)$ is complex-valued in the general case because the values of β_i (and hence of c_i) are complex constants. Thus, the solution (25) is, in general, a complex function. Consequently, there is a problem in selecting the real solutions from the complex solutions. It turns out that we can obtain the real solutions by means of restriction of arbitrariness in the choice of the constants β_i. We have succeeded in finding these restrictions.

4. Real solutions associated with the bound state spectrum. The features of the solutions associated with bound state spectrum can be shown by considering the two-soliton solution for which $M = 2$, $N = 0$. The solution (25) can be obtained through (20), (22),

In Appendix A it is proved that the constants c_i can be only real ones. Moreover, the signs of $\alpha_i = c_i / |c_i|$ can independently take the values ± 1, i.e. we have four variants, namely $\alpha_1 = \alpha_2 = 1$, $\alpha_1 = \alpha_2 = -1$, $\alpha_1 = -\alpha_2 = 1$ and $\alpha_1 = -\alpha_2 = -1$. Note that in [15] only the first two variants are observed. The standard soliton solution for which $\alpha_1 = \alpha_2 = 1$ and the singular soliton solutions for which $\alpha_1 = \alpha_2 = -1$, $\alpha_1 = -\alpha_2 = 1$ and $\alpha_1 = -\alpha_2 = -1$, are obtained by means of the relation (25)

$$U(X,T) = W(X,T)_X = 6\frac{\partial^2}{\partial X^2}\ln(F) = 6\frac{\partial^2}{\partial X^2}\ln(G_i), \qquad (26)$$

where G_i are defined by (A.6) – (A.9).

For $N \geq 3$ we give the conditions without proof. All the constants c_i are to be real and the signs of $\alpha_i = c_i / |c_i|$ can equal to ± 1 independently of each other.

5. Real solutions associated with the continuous spectrum. We study the multi-mode solutions for $M = 0$ and $N = 1, 3$, while for $N \geq 4$ all formulas can easily be obtained by means of a generalization of these examples.

5.1. The one-mode solution. In order to obtain the one-mode solution of the VPE (1) we need first to calculate the 2×2 matrix $M(X,T)$ according to (16) with $M = 0$ and $N = 1$. From (19), (23) we find

$$\det M(X,T) = (1 + c_1 \exp(-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T))^2, \quad c_1 = \frac{i\beta_1}{2\sqrt{3}\xi_1}. \qquad (27)$$

As it has been already noted, the singularity functions in the form (8) with $N = 1$ give rise to a single frequency for the continuous part of the spectral data. Hence, the expression (27), having been substituted into the concise formula (25), must provide us with the one-mode solution.

The condition that W_X is real requires a restriction on the constant β_1 (if the constant ξ_1 is arbitrary but real). We have succeeded in obtaining this restriction (see Appendix B), namely that the constant c_1, which in general is the complex-valued one $c_1 = |c_1|\exp(i\chi_1)$, should possess the unity modulus $|c_1| = 1$, while the arbitrary real constant χ_1 defines an initial shift of solution $X_1 = \chi_1 / (\sqrt{3}\xi_1)$ so that

$$\det M(X,T) = \left[1 + \exp\left(-i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1}\right)\right]^2. \qquad (28)$$

The final result for one mode of the continuous spectrum is the solution (25) with (28), namely,

$$W(X,T) = -3\sqrt{3}\xi_1 \tan\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1}\right) + \text{const.} \qquad (29)$$

The corresponding solution for $U = W_X$ was obtained recently by other methods, for example, by the sine–cosine method [16], the (G'/G)–expansion method [9], and the extended tanh–function method [16, 17, 18]. However, only the approach developed here and the solution in the form (15), (16) enable us to study the interaction of solitons and periodic waves.

5.2. The three-mode solution. For $N = 3$ and $M = 0$ in the relationship (21) with (23), we write $c_i = |c_i| \exp(i\chi_i)$. Then the arguments χ_i determine the initial phase shifts of modes $X_i = \chi_i / (\sqrt{3}\xi_i)$. As is proved in Appendix B, the conditions on the constants c_1 (or the same on β_1) are

$$|c_1| = 1/\sqrt{b_{12}b_{13}}, \qquad |c_2| = 1/\sqrt{b_{12}b_{23}}, \qquad |c_3| = 1/\sqrt{b_{13}b_{23}}. \tag{30}$$

Hence, the three-mode solution is the relation (25) with

$$F(X,T) = 1 + \frac{1}{\sqrt{b_{12}b_{13}}}(q_1 + q_2q_3) + \frac{1}{\sqrt{b_{12}b_{23}}}(q_2 + q_1q_3) + \frac{1}{\sqrt{b_{13}b_{23}}}(q_3 + q_1q_2) + q_1q_2q_3. \tag{31}$$

6. Real soliton and multi-mode solutions. In this subsection we will consider the general case, when both the bound state spectrum and the continuous spectrum are taken into account in the associated spectral problem. We will find the conditions on c_i for real solutions of the VPE. To obtain the solution, we need to know the function F (see (19)–(24)).

Let the indexes i, i' be related to the values involved in the bound state spectrum for which $(i,i') \le M$, while the indexes j, j' are related to the values involved in the continuous part of the spectral data for which $M < (j, j') \le M + N$.

6.1. The interaction of a soliton with one-mode wave. The interaction of a standard soliton with periodic one-mode wave can be described by means of the relations (20) with q_i and b_{12} as in (24). First, we emphasize that the soliton and one-mode wave (29) propagate in opposite directions. The soliton propagates in the positive direction of the X–axis, while the one-mode wave (29) propagates in the negative direction of the X–axis.

Here we restrict ourselves to the simplest case $b_{12}c_1c_2 = 1$ that describes the interaction of a standard soliton with a one-mode wave. As follows immediately from Appendix B, for real solutions (25) we have

$$F(X,T) = 1 + \frac{1}{\sqrt{b_{12}}}q_1 + \frac{1}{\sqrt{b_{12}}}q_2 + q_1q_2. \tag{32}$$

There is an exceptional case at $\xi_1 = \xi_2$. Then we have $b_{12} = 1$, and $F = (1 + q_1)(1 + q_2)$. Consequently, the solution (25) is reduced to the relation

$$W = W_1 + W_2 = 3\sqrt{3}\xi_1 \tanh\left(\frac{\sqrt{3}}{2}\xi_1 (X - X_1) - \frac{T}{2\sqrt{3}\xi_1} \right)$$
$$- 3\sqrt{3}\xi_1 \tan\left(\frac{\sqrt{3}}{2}\xi_1 (X - X_0) + \frac{T}{2\sqrt{3}\xi_1} \right) + \text{const.} \quad (33)$$

Here W_1 is the one-soliton solution and W_2 is the solution (29) associated with one mode in the continuous part of the spectral data. The relationship $W = W_1 + W_2$ is easily verified also by direct substitution into Eq. (1). The two waves W_1 and W_2 propagate in different directions with the same speed without change of wave profile.

6.2. Real solutions for M solitons and the N–mode wave. The interaction of M solitons and the N–mode wave (25) can be obtained by means of the function $F(X,T)$ with restrictions (B.6) given in Appendix B, namely

$$c_i = \pm 1 / \sqrt{\prod_{\substack{j=1 \\ j\neq i}}^{M+N} b_{ij}}, \quad b_{ij} = b_{ji}, \quad i = 1,...,M+N, \quad (34)$$

and with the retention of the phase shifts X_i in the quantities q_i (B.2). The signs for c_i in (34) can be chosen independently of each other. If the index i in (34) is connected with the continuous part of the spectral data ($M < i \leq M+N$), then the solutions generated by `plus' and `minus' signs in (34) are different only in the phase shifts. However, for the index i from the bound state spectrum ($i \leq M$), the solutions have different forms of function dependencies. Here it is relevant to remember that there are standard soliton solutions and singular soliton solutions generated by different signs in the constants c_i (34).

The solution will contain $(M+N)$ real constants ξ_i for determining the values b_{ij} and $(M+N)$ real constants X_i to define the phase shifts.

7. Conclusion. The procedure for finding the solutions of the Vakhnenko–Parkes equation by means of the inverse scattering method is described. Both the bound state spectrum and the continuous spectrum are taken into account in the associated eigenvalue problem. The special form of the singularity functions enables us to obtain the multi-mode solutions. Sufficient conditions have been proved in order that the solutions become real functions. Finally we studied the interaction of the solitons and the multi-mode wave.

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Appendix A.

Here we consider the conditions on signs for the constants c_i under the interaction of two solitons ($M = 2$, $N = 0$). We start with the relationship (20), (22)

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2. \tag{A.1}$$

Let us present the constants c_i in the form

$$c_i = \alpha_i \, | c_i | \exp(i\chi_i) = b_{12}^{-1/2} \exp\left(-\sqrt{3}\xi_i X_i + i\sigma_i\right), \quad \sigma_i = \chi_i + \pi(1-\alpha_i)/2. \tag{A.2}$$

All new constants χ_i and $X_i = -\ln(| c_i \sqrt{b_{12}} |)/(\sqrt{3}\xi_i)$ are real. We assume that $-\pi/2 < \chi_i \le \pi/2$, then the values α_i retain the signs of the constants $\mathrm{Re}(c_i)$, i.e. $\alpha_i = \mathrm{Re}(c_i)/|\mathrm{Re}(c_i)|$. It is convenient for analyzing to rewrite (A.1) in the form

$$F = 2\exp\left(\theta_1 + \theta_2 + \frac{i}{2}(\sigma_1 + \sigma_2)\right) G. \tag{A.3}$$

with

$$G = \cosh\left(\theta_1 + \theta_2 + \frac{i}{2}(\sigma_1 + \sigma_2)\right) + b_{12}^{-1/2} \cosh\left(\theta_1 - \theta_2 + \frac{i}{2}(\sigma_1 - \sigma_2)\right), \tag{A.4}$$

$$2\theta_i = \sqrt{3}\xi_i (X - X_i) - (\sqrt{3}\xi_i)^{-1} T.$$

It is easily seen that only G defines the solution, since $\frac{\partial^2}{\partial X^2}\ln(F) = \frac{\partial^2}{\partial X^2}\ln(G)$, while the conditions that the function G is real are as follows:

$$\chi_i = 0, \quad \sigma_i + \sigma_2 = 2\pi k_1, \quad \sigma_i - \sigma_2 = 2\pi k_2 \tag{A.5}$$

with $k_i = 0, 1$. These restrictions (A.5) lead to the requirements $\alpha_1 = \pm 1$, $\alpha_2 = \pm 1$, independently of each other, and $\chi_i = 0$. Then the function F has the following forms:

(i) for $\alpha_1 = \alpha_2 = 1$

$$F = 2\exp(\theta_1 + \theta_2) G_1, \quad G_1 = \cosh(\theta_1 + \theta_2) + b_{12}^{-1/2} \cosh(\theta_1 - \theta_2); \tag{A.6}$$

(ii) for $\alpha_1 = \alpha_2 = -1$

$$F = 2\exp(\theta_1 + \theta_2) G_2, \quad G_2 = \cosh(\theta_1 + \theta_2) - b_{12}^{-1/2} \cosh(\theta_1 - \theta_2); \tag{A.7}$$

(iii) for $\alpha_1 = -\alpha_2 = 1$

$$F = 2\exp(\theta_1 + \theta_2) G_3, \quad G_3 = -\sinh(\theta_1 + \theta_2) + b_{12}^{-1/2} \sinh(\theta_1 - \theta_2); \tag{A.8}$$

(iv) for $\alpha_1 = -\alpha_2 = -1$

$$F = 2\exp(\theta_1 + \theta_2) G_4, \quad G_4 = -\sinh(\theta_1 + \theta_2) - b_{12}^{-1/2} \sinh(\theta_1 - \theta_2). \tag{A.9}$$

Hence, the standard soliton solution that follows from (A.6) and the singular soliton solutions that follow from (A.7)–(A.9) are the real functions

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(G_i). \tag{A.10}$$

Now we rewrite the restrictions in somewhat different form. By retaining the values of the phaseshifts X_i in the quantities q_i, we require

$$c_1 = \pm\sqrt{b_{12}}, \quad c_2 = \pm\sqrt{b_{12}}, \tag{A.11}$$

where the signs are independent of each other. Note that for this case there are two arbitrary real constants ξ_i, and two arbitrary real constants X_i ($i = 1,2$).

The notation in (A.6)–(A.9) shows that the solution is defined by two combinations of the spectral parameters, namely $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$, but not three values ξ_1, ξ_2, $\xi_1 + \xi_2$ as it may appear from (A.1).

The foregoing proof points to a way for finding the restrictions for any M with $N = 0$. Here it should be underlined that only at real c_i with any sign of $\alpha_i = c_i / |c_i|$, the soliton (or singular soliton) solutions are determined by a real function. The conditions on the constants c_i are as follows:

$$c_i = \pm 1 / \sqrt{\prod_{\substack{j=1 \\ j\neq i}}^{M} b_{12}} \quad i = 1,..., M, \tag{A.12}$$

with the retention of the phase shifts X_i in the quantities q_i. The signs for c_i are independent of each other. The solution will contain the M real constants ξ_i for determining the values b_{ij} and the M real constants X_i to define the phase shifts.

Appendix B.

Here we will obtain the restrictions on the constants c_i for real solutions, in the general case, taking into account the spectral data from both the bound state spectrum and the continuous spectrum. All features are inherent in the case $M + N = 3$ considered here as an example. To find the solution by means of the inverse scattering method, one needs to know the function (21)

$$\begin{aligned} F = 1 &+ c_1q_1 + c_2q_2 + c_3q_3 + b_{12}c_1c_2q_1q_2 + b_{13}c_1c_3q_1q_3 \\ &+ b_{23}c_2c_3q_2q_3 + b_{12}b_{13}b_{23}c_1c_2c_3q_1q_2q_3. \end{aligned} \tag{B.1}$$

For convenience we rewrite the variables q_i in the somewhat different form

$$\begin{aligned} &q_i \exp(2\theta_i), \quad q_j \exp(i2\theta_j), \quad 2\theta_i = \sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1/2}T, \\ &2\theta_j = -\sqrt{3}\xi_j(X - X_j) - (\sqrt{3}\xi_j)^{-1/2}T, \end{aligned} \tag{B.2}$$

The phase shifts X_i are the arbitrary real constants. The values b_{ij} in (B.1) are as in (24). Note that $b_{ii'}$, $b_{jj'}$ are real values, and $b_{ij}^* = 1/b_{ij}$. Without loss of generality, we will consider one set of values M,N, for example $M = 1$, $N = 2$. Now we will show that the restrictions

$$c_1 = \pm 1/\sqrt{b_{12}b_{13}}, \quad c_2 = \pm 1/\sqrt{b_{12}b_{23}}, \quad c_3 = \pm 1/\sqrt{b_{13}b_{23}} \qquad \text{(B.3)}$$

(with b_{ij} determined by (24)) are sufficient in order to obtain the real solutions.

For definiteness, we assume that $\sqrt{b_{ij}}$ is a root of an equation $x^2 = b_{ij}$ with $-\pi/2 < \arg\sqrt{b_{ij}} \le \pi/2$. Let us rewrite the relations (B.3) in the form $c_i = \alpha_i / \prod_{\substack{j=1 \\ j\neq i}}^{3} \sqrt{b_{ij}}$ where $\alpha_i = \pm 1$. It is evident that we can always attain $\alpha_2 = \alpha_3 = 1$ by choosing the phase shifts X_2, X_3, while we need to consider the two cases $\alpha_1 = \pm 1$. By defining $\sigma = (1-\alpha_1)/2$, we can rewrite the auxiliary function F from (B.1) in the form

$$\begin{aligned}
&F(X,T) = 2Ge^{i\pi\sigma}(b_{12}b_{13})^{-1/4}\exp(\theta_1 + i\pi\sigma/2 + i\theta_2 + i\theta_3), \\
&Ge^{i\pi\sigma} = [(b_{12}b_{13})^{1/4}\exp(-i\theta_1 + \pi\sigma/2 + \theta_2 + \theta_3) \\
&\quad + (b_{12}b_{13})^{-1/4}\exp(-i\theta_1 + \pi\sigma/2 - \theta_2 - \theta_3)] \\
&\quad + (b_{23})^{-1/2}[(b_{13}/b_{12})^{1/4}\exp(i\theta_1 - \pi\sigma/2 + \theta_2 - \theta_3) \\
&\quad + (b_{13}/b_{12})^{-1/4}\exp(-i\theta_1 + \pi\sigma/2 + \theta_2 - \theta_3)] \\
&\quad + (b_{23})^{-1/2}[(b_{12}/b_{13})^{1/4}\exp(i\theta_1 - \pi\sigma/2 - \theta_2 + \theta_3) \\
&\quad + (b_{12}/b_{13})^{-1/4}\exp(-i\theta_1 + \pi\sigma/2 - \theta_2 + \theta_3)].
\end{aligned} \qquad \text{(B.4)}$$

Since b_{23} is real, and $b_{1j}^* = 1/b_{1j}$ for $j = 2, 3$, it is evident that $G^* = G$, i.e. the variable G in the solution is a real-valued function. Hence, the solution of the VPE

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(F) = 6\frac{\partial^2}{\partial X^2}\ln(G) \qquad \text{(B.5)}$$

represents a real quantity.

Using this example, one can prove without difficulty that the procedure considered above can be extended to any M,N with restrictions

$$c_i = \pm 1/\sqrt{\prod_{\substack{j=1 \\ j\neq i}}^{M+N} b_{ij}}, \quad b_{ij} = b_{ji}, \quad i = 1,\ldots,M+N, \qquad \text{(B.6)}$$

while the quantities q_i retain the phase shifts X_i (see (B.2)). The signs in (B.6) can be chosen independently of each other. For interaction of M solitons and the N–mode wave there are $(M+N)$ real constants ξ_i and $(M+N)$ real constants X_i.

Note that the restrictions (B.6) are sufficient conditions in order that the solution of the VPE becomes real.

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