ON INFINITESIMAL MG–DEFORMATIONS OF A SURFACE OF POSITIVE GAUSSIAN CURVATURE WITH STATIONARITY OF NORMAL CURVATURE ALONG THE BOUNDARY

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Investigation of infinitesimal deformations with some additional conditions is one of the directions of geometry, in present time, for example, the AG–deformations, when Grassmanian image and an area element $d\theta$ of a surface are kept ($\delta\vec{n} = 0, \quad \delta(d\theta) = 0$), the ARG–deformations, when Grassmanian image of a surface is kept, and an area element $d\theta$ of a surface is changed by the recurrent law ($\delta\vec{n} = 0, \quad \delta(d\theta) = 2H \cdot \chi \cdot c \cdot d\theta$), and others. See [5] for more details.

The aim of this paper is to research the infinitesimal deformations with conditions $\delta K = \sigma, \quad \delta\vec{n} = 0$. This deformations, named the MG–deformations, give increment of Gaussian curvature as the function σ on a surface and keep Grassmanian image of the surface.

Remark that if $\sigma \equiv 0$, and $\chi \equiv 0$, the AG–deformations, the ARG–deformations and the MG–deformations coincide.

The research is done in three-dimensional Euclidian space E^3.

A simple connected surface *S* with Gaussian curvature $K \geq k_0 > 0$, $k_0 = \text{const}$, and with the boundary is subjected to deformation. The surface *S* is given as the position vector $\vec{r} = \vec{r}(u, v) \in D_{3,p}, p > 2$, $(u, v) \in \Omega$, Ω is a flat simple connected region with the boundary $\Gamma \in C^1_\mu, 0 < \mu \leq 1$. A normal curvature is stationary along the boundary. We assume the field of deformation is a function of $D_{2,p}, p > 2$, class, $\sigma \in D_{1,p}, p > 2$.

In the beginning of the article we give the definition of the infinitesimal MG–deformation. Then we get the simultaneous equations, described the infinitesimal MG–deformation of the surface. Then we transform the boundary condition. After that we convert this boundary value problem to complex form, that leads us to the investigation of the generalized Riemann–Gilbert boundary value problem, and to the calculation of the index. And finally, the decision problem of this boundary value problem is researched by method from [1], that gives the main result of this work.

Theorem. *Let S is a simple connected surface of* $D_{3,p}, p > 2$*, class. This surface map into the flat region* Ω *one-to-one,* Ω *has the boundary* $\Gamma \in C^1_\mu, 0 < \mu \leq 1$. *The Gaussian curvature of the surface* $K \geq k_0 > 0$, $k_0 = \text{const}$. *Let S is subjected to the infinitesimal MG–deformation with stationarity of a normal curvature along the boundary.*

Then:

a) *with* $\sigma \equiv 0$ *there exist the trivial infinitesimal MG–deformation only;*
b) *with* $\sigma \not\equiv 0$ *there exist the unique infinitesimal MG–deformation if and only if function σ obey three conditions:*

$$\int_{\Gamma} \gamma(t) w_j'(t) \lambda(t) dt = 0, \quad (j = 1, 2, 3),$$

where w_1', w_2', w_3' *are a full set of the solutions of the conjugate homogeneous problem* $\overset{\circ}{A}'$.

These results are theoretical. The MG–deformations have not been considered in literature.

1. The concept of the infinitesimal MG–deformation. Let *S* is a simple connected surface in E^3. *S* is given by position vector $\vec{r} = \vec{r}(u, v) \in D_{3,p}, p > 2$, $(u, v) \in \Omega$, where Ω is a flat simple connected region.

We consider a deformation S_t, $t \in (-t_0, t_0)$, $t_0 > 0$, of the surface *S*, given by equation $\vec{r}_t = \vec{R}(u, v, t)$, $(u, v) \in \Omega$, where $\vec{R}(u, v, t)$ is the function, which belongs to the class $D_{3,p}, p > 2$, on parameters u, v, and to the class C^2 on parameter *t*, $\vec{R}(u, v, 0) \equiv \vec{r}(u, v)$.

Let $a(u, v)$ is any function on the surface *S*. It will become a function $A(u, v, t)$, after the deformation, and $A(u, v, 0) \equiv a(u, v)$.

The function $\delta a = \left.\frac{\partial A}{\partial t}\right|_{t=0}$ is called *variation* of the function *a* of the surface *S* with deformation S_t. We will denote the vector field of the deformation $\left.\frac{\partial \vec{R}}{\partial t}\right|_{t=0} = \delta\vec{r}$ as $\vec{y}$. Throughout this work we assume $\vec{y} = \vec{y}(u, v) \in D_{2,p}, \ p > 2$.

Two deformations are called *equivalent*, if their deformation fields are equal. We call each class of equivalent deformations the *infinitesimal deformation* of the surface *S*.

The *infinitesimal MG–deformation* is infinitesimal deformation of the surface *S*, with conditions:

$$\delta K = \sigma, \quad \delta\vec{n} = 0, \tag{1}$$

where δK is the variation of Gaussian curvature and $\delta\vec{n}$ is the variation of the unit normal vector, *σ* is a given function on the surface *S*.

If the field of the deformation is $\vec{y} = \overrightarrow{const}$, i. e. it correspond to infinitesimal parallel displacement of the surface *S* in the space, then such infinitesimal MG–deformation is called *trivial*.

Remark. The infinitesimal deformation with condition $\delta\vec{n} = 0$ is named G–deformation [3]. The condition $\delta K = \sigma$ is connected with Minkowsky problem [4], and so we have such name of deformation (MG–deformation).

2. The simultaneous equations for the infinitesimal MG–deformation. The consequence from the condition $\delta\vec{n} = 0$ is that vectors $\partial_1\vec{r} = \frac{\partial\vec{r}}{\partial u}$, $\partial_2\vec{r} = \frac{\partial\vec{r}}{\partial v}$, $\partial_1\vec{y} = \frac{\partial\vec{y}}{\partial u}$, $\partial_2\vec{y} = \frac{\partial\vec{y}}{\partial v}$ are coplanar. Therefore we can decompose $\partial_1\vec{y}$, $\partial_2\vec{y}$ on

$$\partial_1\vec{r},\ \partial_2\vec{r}:\quad u^2 \tag{2}$$

where α_j^k – some scalar functions of u, v. From (2) we get

$$\begin{cases}\partial_{12}\vec{y} = \partial_2\alpha_1^1\partial_1\vec{r} + \alpha_1^1\partial_{12}\vec{r} + \partial_2\alpha_1^2\partial_2\vec{r} + \alpha_1^2\partial_{22}\vec{r},\\ \partial_{21}\vec{y} = \partial_1\alpha_2^1\partial_1\vec{r} + \alpha_2^1\partial_{11}\vec{r} + \partial_1\alpha_2^2\partial_2\vec{r} + \alpha_2^2\partial_{21}\vec{r}.\end{cases} \tag{3}$$

Using Gauss formulas $\partial_{jk}\vec{r} = \Gamma_{jk}^l\partial_l\vec{r} + b_{jk}\vec{n};\ j,k = 1,2$, where b_{jk} – coefficients of the second quadratic form of the initial surface, Γ_{jk}^l – Christoffel symbols, we get the set of equations:

$$\begin{cases}\partial_2\alpha_1^1 + \alpha_1^1\Gamma_{12}^1 + \alpha_1^2\Gamma_{22}^1 = \partial_1\alpha_2^1 + \alpha_2^1\Gamma_{11}^1 + \alpha_2^2\Gamma_{21}^1,\\ \partial_2\alpha_1^2 + \alpha_1^1\Gamma_{12}^2 + \alpha_1^2\Gamma_{22}^2 = \partial_1\alpha_2^2 + \alpha_2^1\Gamma_{11}^2 + \alpha_2^2\Gamma_{21}^2,\\ \alpha_1^1 b_{12} + \alpha_1^2 b_{22} = \alpha_2^1 b_{11} + \alpha_2^2 b_{21}.\end{cases} \tag{4}$$

Since Gaussian curvature of the initial surface *S*: $K \ge k_0 > 0$, $k_0 = \text{const}$, then we can introduce the conjugate isometric coordinate system [1, p. 92, 93]. In this system we have $b_{11} = b_{22} \ne 0$, $b_{12} = 0$, therefore $\alpha_1^2 = \alpha_2^1$. We transform the set of equation (4) to the system:

$$\begin{cases}\partial_2\alpha_1^1 - \partial_1\alpha_1^2 = (\alpha_2^2 - \alpha_1^1)\Gamma_{21}^1 + \alpha_1^2(\Gamma_{11}^1 - \Gamma_{22}^1),\\ \partial_2\alpha_1^2 - \partial_1\alpha_2^2 = (\alpha_2^2 - \alpha_1^1)\Gamma_{12}^2 + \alpha_1^2(\Gamma_{11}^2 - \Gamma_{22}^2).\end{cases} \tag{5}$$

Then, using the method from [4], we introduce the notation:

$U = \frac{1}{2}(\alpha_2^2 - \alpha_1^1)$, $V = \alpha_1^2$, $\Pi = \frac{1}{2}(\alpha_2^2 + \alpha_1^1)$, therefore $\alpha_1^1 = \Pi - U$, $\alpha_2^2 = \Pi + U$.

Equations (5) are transformed to:

$$\begin{cases}\partial_1 U - \partial_2 V + 2\Gamma_{12}^2 U + (\Gamma_{11}^2 - \Gamma_{22}^2)V = -\partial_1\Pi,\\ \partial_2 U + \partial_1 V + 2\Gamma_{21}^1 U + (\Gamma_{11}^1 - \Gamma_{22}^1)V = \partial_2\Pi.\end{cases} \tag{6}$$

Thus, from the condition $\delta\vec{n} = 0$, we get the simultaneous equations (6).

Now, we consider the condition $\delta K = \sigma$.

Let $g = g_{11}g_{22} - g_{12}^2$ is the discriminant of the first quadratic form, and $b = b_{11}b_{22} - b_{12}^2$ is the discriminant of the second quadratic form, then $K = \frac{b}{g}$, and $\delta\left(\frac{b}{g}\right) = \sigma$, therefore

$$\delta b - K \cdot \delta g = g\sigma . \tag{7}$$

We will take the variations of *g* and *b*:

$$\begin{aligned} \delta g &= \delta g_{11} \cdot g_{22} + g_{11} \cdot \delta g_{22} - 2g_{12} \cdot \delta g_{12}, \\ \delta b &= \delta b_{11} \cdot b_{22} + b_{11} \cdot \delta b_{22} - 2b_{12} \cdot \delta b_{12}. \end{aligned} \tag{8}$$

First, we will calculate the variations δg_{jk} and δb_{jk}, $j,k = 1,2$, expressing them by α_1^1, α_1^2, α_2^2 and the coefficients of the first and the second quadratic forms of *S*. Taking account of $\delta\vec{r} = \vec{y}$, $\delta\vec{n} = 0$, we have:

$$\begin{aligned} \delta g_{11} &= \delta(\partial_1\vec{r}, \partial_1\vec{r}) = 2\alpha_1^1 g_{11} + 2\alpha_1^2 g_{21}, \\ \delta g_{12} &= \delta(\partial_1\vec{r}, \partial_2\vec{r}) = \alpha_1^1 g_{12} + \alpha_1^2 g_{22} + \alpha_1^2 g_{11} + \alpha_2^2 g_{21}, \\ \delta g_{22} &= \delta(\partial_2\vec{r}, \partial_2\vec{r}) = 2\alpha_1^2 g_{12} + 2\alpha_2^2 g_{22}. \end{aligned} \tag{9}$$

Now, we differentiate with respect to u the first of formulas (2), and we differentiate with respect to v the second of formulas (2). Using getting equalities and the first equality of the system (3), we find:

$$\begin{aligned} \delta b_{11} &= (\delta(\partial_{11}\vec{r}), \vec{n}) + (\partial_{11}\vec{r}, \delta\vec{n}) = \alpha_1^1 b_{11} + \alpha_1^2 b_{12}, \\ \delta b_{12} &= (\delta(\partial_{12}\vec{r}), \vec{n}) + (\partial_{12}\vec{r}, \delta\vec{n}) = \alpha_1^1 b_{12} + \alpha_1^2 b_{22}, \\ \delta b_{22} &= (\delta(\partial_{22}\vec{r}), \vec{n}) + (\partial_{22}\vec{r}, \delta\vec{n}) = \alpha_1^2 b_{12} + \alpha_2^2 b_{22}. \end{aligned}$$

Since $b_{11} = b_{22} \neq 0$, $b_{12} = 0$, then $\delta b_{11} = \alpha_1^1 b_{11}$, $\delta b_{12} = \alpha_1^2 b_{11}$, $\delta b_{22} = \alpha_2^2 b_{11}$.

Thus, δb_{jk} are described by the formulas:

$$\delta b_{11} = \alpha_1^1 b_{11}, \ \delta b_{12} = \alpha_1^2 b_{11}, \ \delta b_{22} = \alpha_2^2 b_{11}. \tag{10}$$

Substituting (10) and (9) into (8), and then substituting the result into (7), we get:

$$\begin{aligned} & b_{11}b_{11}(\alpha_1^1 + \alpha_2^2) - 2Kg_{22}(\alpha_1^1 g_{11} + \alpha_1^2 g_{21}) - 2Kg_{11}(\alpha_1^2 g_{12} + \alpha_2^2 g_{22}) + \\ & +2Kg_{12}(\alpha_1^1 g_{12} + \alpha_1^2 g_{22} + \alpha_1^2 g_{11} + \alpha_2^2 g_{21}) = g\sigma. \end{aligned}$$

Therefore $(\alpha_1^1 + \alpha_2^2)(b_{11}b_{11} - 2Kg) = g\sigma$, and

$$\alpha_1^1 + \alpha_2^2 = -\frac{\sigma}{K}. \tag{11}$$

Introducing the notation $\Pi = \frac{1}{2}(\alpha_2^2 + \alpha_1^1)$, we substitute it into (11) and get:

$$\Pi = -\frac{\sigma}{2K}. \tag{12}$$

Thus, we have shown that the infinitesimal MG–deformation of the surface of Gaussian curvature $K \geq k_0 > 0$, $k_0 = \text{const}$, implies the system:

$$\begin{cases} \partial_1 U - \partial_2 V + 2\Gamma_{12}^2 U + (\Gamma_{11}^2 - \Gamma_{22}^2)V = -\partial_1 \Pi, \\ \partial_2 U + \partial_1 V + 2\Gamma_{21}^1 U + (\Gamma_{11}^1 - \Gamma_{22}^1)V = \partial_2 \Pi, \\ \Pi = -\dfrac{\sigma}{2K}. \end{cases} \tag{13}$$

The function Π is described by the third formula of the system (13), where *K* and *σ* are known functions. So we can substitute $-\dfrac{\sigma}{2K}$ into the others formulas of (13), so we get the system of two equations with two unknown *U* and *V*:

$$\begin{cases} \partial_1 U - \partial_2 V + 2\Gamma_{12}^2 U + (\Gamma_{11}^2 - \Gamma_{22}^2)V = \partial_1\left(\dfrac{\sigma}{2K}\right), \\ \partial_2 U + \partial_1 V + 2\Gamma_{21}^1 U + (\Gamma_{11}^1 - \Gamma_{22}^1)V = -\partial_2\left(\dfrac{\sigma}{2K}\right). \end{cases} \tag{14}$$

Thus, the searching for the field $\vec{y}$ is reduced to following operations. First we have to find *U* and *V*. If we know *U* and *V*, we will get α_1^1, α_1^2, α_2^2 by using formulas $\alpha_1^1 = \Pi - U$, $V = \alpha_1^2$, $\alpha_2^2 = \Pi + U$ and (12). Then from (2), we get $\partial_1 \vec{y}$, $\partial_2 \vec{y}$. The surface *S* is the simple connected one, so we find the vector field of the MG–deformation by integration of $d\vec{y} = \partial_1 \vec{y} du + \partial_2 \vec{y} dv$ with accuracy to a constant vector.

3. The transformation of the boundary condition. The normal curvature is stationary along the boundary, i. e. $\delta k_n = 0$. It is known, that normal curvature describes by formula $k_n = \dfrac{II}{I}$, where *I* and *II* – the first and the second quadratic forms of the surface, so the boundary condition is:

$$\frac{\delta II \cdot I - II \cdot \delta I}{I^2} = 0.$$

Therefore

$$\delta II \cdot I - II \cdot \delta I = 0. \tag{15}$$

From well-known formulas $I = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2$ and $II = b_{11} du^2 + 2b_{12} du dv + b_{22} dv^2$ follow $\delta I = \delta g_{11} du^2 + 2\delta g_{12} du dv + \delta g_{22} dv^2$ and $\delta II = \delta b_{11} du^2 + 2\delta b_{12} du dv + \delta b_{22} dv^2$, then the equation (15) takes the form:

$$(\delta b_{11}du^2+2\delta b_{12}dudv+\delta b_{22}dv^2)\cdot(g_{11}du^2+2g_{12}dudv+g_{22}dv^2)-$$
$$-(b_{11}du^2+2b_{12}dudv+b_{22}dv^2)\cdot(\delta g_{11}du^2+2\delta g_{12}dudv+\delta g_{22}dv^2)=0. \quad (16)$$

Using the division of the equality (16) by $ds^4>0$ and taking account of $b_{11}=b_{22}\neq 0$, $b_{12}=0$, we have:

$$(\alpha_1^1 b_{11}\dot u^2+2\alpha_1^2 b_{11}\dot u\dot v+\alpha_2^2 b_{11}\dot v^2)(g_{11}\dot u^2+2g_{12}\dot u\dot v+g_{22}\dot v^2)-$$
$$-b_{11}(\dot u^2+\dot v^2)(\delta g_{11}\dot u^2+2\delta g_{12}\dot u\dot v+\delta g_{22}\dot v^2)=0,$$

where $\dot u=\frac{du}{ds}$, $\dot v=\frac{dv}{ds}$. Dividing the last equation by b_{11}, we have:

$$(\alpha_1^1\dot u^2+2\alpha_1^2\dot u\dot v+\alpha_2^2\dot v^2)(g_{11}\dot u^2+2g_{12}\dot u\dot v+g_{22}\dot v^2)-$$
$$-(\dot u^2+\dot v^2)(\delta g_{11}\dot u^2+2\delta g_{12}\dot u\dot v+\delta g_{22}\dot v^2)=0.$$

Taking into account (9), we regroup this relation and finally have:

$$\alpha_1^1(-g_{11}\dot u^4+(g_{22}-2g_{11})\dot u^2\dot v^2-2g_{12}\dot u\dot v^3)+$$
$$+\alpha_1^2(-2g_{12}\dot u^4-2g_{22}\dot u^3\dot v-2g_{11}\dot u\dot v^3-2g_{12}\dot v^4)+$$
$$+\alpha_2^2(-g_{22}\dot v^4+(g_{11}-2g_{22})\dot u^2\dot v^2-2g_{12}\dot u^3\dot v)=0.$$

We introduce the notation:

$$a_1=-g_{11}\dot u^4+(g_{22}-2g_{11})\dot u^2\dot v^2-2g_{12}\dot u\dot v^3,$$
$$a_2=-2g_{12}\dot u^4-2g_{22}\dot u^3\dot v-2g_{11}\dot u\dot v^3-2g_{12}\dot v^4,$$
$$a_3=-g_{22}\dot v^4+(g_{11}-2g_{22})\dot u^2\dot v^2-2g_{12}\dot u^3\dot v.$$

Thus, we can write the boundary condition $\delta k_n=0$ as:

$$\alpha_1^1 a_1+\alpha_1^2 a_2+\alpha_2^2 a_3=0, \quad (17)$$

where a_1, a_2, a_3 are known functions, $\alpha_1^1, \alpha_1^2, \alpha_2^2$ are unknown functions.

We substitute $\alpha_1^1=\Pi-U$, $\alpha_1^2=V$, $\alpha_2^2=\Pi+U$ in (17), and we obtain $U(a_3-a_1)+Va_2=-\Pi(a_3+a_1)$, taking into account (12), we get:

$$U(a_3-a_1)+Va_2=\frac{\sigma}{2K}(a_3+a_1). \quad (18)$$

Thus, the searching of the infinitesimal MG–deformation of the surface *S* leads us to the boundary value problem (14) – (18).

4. The complex form of the boundary value problem. We introduce the function $w(z)=U+iV$, where $z=u+iv$, $(u,v)\in\Omega$, following the method from [1, p. 111], and we describe the set of equations (14) as the one complex equation

$$\partial_{\bar z}w+A_1w+B_1\bar w=\frac{1}{2}\partial_z\left(\frac{\sigma}{K}\right), \quad (19)$$

where $A_1 = \frac{1}{4}(\Gamma^1_{11} - \Gamma^1_{22} + 2\Gamma^2_{12}) - \frac{i}{4}(\Gamma^2_{11} - \Gamma^2_{22} - 2\Gamma^1_{21})$,

$B_1 = \frac{1}{4}(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}) + \frac{i}{4}(\Gamma^2_{11} - \Gamma^2_{22} + 2\Gamma^1_{12})$,

$\partial_{\bar z} w = \frac{1}{2}(\partial_1 w + i\partial_2 w)$, $\partial_z\left(\frac{\sigma}{K}\right) = \frac{1}{2}\left(\partial_1\left(\frac{\sigma}{K}\right) - i\partial_2\left(\frac{\sigma}{K}\right)\right)$.

Let $\lambda = a_3 - a_1 + ia_2$, $w = U + iV$.

$\bar\lambda w = (a_3 - a_1 - ia_2)(U + iV) = (a_3 - a_1)U + iV(a_3 - a_1) - ia_2 U + Va_2$.

$\mathrm{Re}\{\bar\lambda w\} = (a_3 - a_1)U + a_2 V$.

Our boundary condition takes the form:

$$\mathrm{Re}\{\bar\lambda w\} = \frac{\sigma}{2K}(a_3 + a_1). \tag{20}$$

So, we have got the boundary value problem (19) – (20), that is complex form of the problem (14) – (18).

From $\vec r = \vec r(u, v) \in D_{3,p}, p > 2$, and $\sigma \in D_{1,p}, p > 2$, follows that:

1) $A_1, B_1, \frac{1}{2}\partial_z\left(\frac{\sigma}{K}\right) \in L_p, p > 2$;

2) $\frac{\sigma}{2K}(a_3 + a_1) \in C_\nu(\Gamma)$, $0 < \nu \le 1$.

Since we investigate the surface of positive Gaussian curvature with the boundary, and this surface map into the flat region Ω one-to-one, we regard that the boundary of the region Ω is the unite circle, without breakdown of generality, see [1, p. 182]. The function λ is:

$\lambda = a_3 - a_1 + ia_2 =$

$= (\dot u - i\dot v)^2[g_{11}\dot u^2 + 2g_{12}\dot u\dot v + g_{22}\dot v^2 + i(2g_{11}\dot u\dot v - 2g_{22}\dot u\dot v + 2g_{12}(\dot v^2 - \dot u^2)]$.

3) $\lambda \in C_\nu(\Gamma)$, $0 < \nu \le 1$.

Obviously $\lambda \ne 0$, therefore (taking into account [2, p. 231]) we can regard $|\lambda| = 1$.

Thus, all conditions of the generalized Riemann–Gilbert boundary value problem [1, p. 179, 182] for a canonical region are observed. Therefore the problem (16) – (20) is the generalized Riemann–Gilbert boundary value problem.

The index of the function λ is the index of the problem.

5. Proof of the theorem.

We denote

$\lambda_1 = g_{11}\dot u^2 + 2g_{12}\dot u\dot v + g_{22}\dot v^2 + i(2g_{11}\dot u\dot v - 2g_{22}\dot u\dot v + 2g_{12}(\dot v^2 - \dot u^2)$,

$\lambda_2 = (\dot u - i\dot v)^2$.

By the properties of an index, we have $\kappa = \mathrm{Ind}\lambda = \mathrm{Ind}\lambda_1 + \mathrm{Ind}\lambda_2$. Since $\mathrm{Re}\{\lambda_1\} = g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2 > 0$, then $\mathrm{Ind}\lambda_1 = 0$, therefore

$$\kappa = \mathrm{Ind}\lambda = \mathrm{Ind}\lambda_2 = \mathrm{Ind}(\dot{u} - i\dot{v})^2 = 2\mathrm{Ind}(\dot{u} - i\dot{v}).$$

Since we regard that the boundary of the region Ω is a unit circle, then $u = \cos\phi,\ \ v = \sin\phi,\ 0 \le \phi \le 2\pi$, therefore $\dot{u} = -\sin\phi,\ \ \dot{v} = \cos\phi$.

Using formula for calculation of the index [2, p. 96], we have:

$$\kappa = \mathrm{Ind}\lambda = 2\mathrm{Ind}(\dot{u} - i\dot{v}) =$$

$$= 2\frac{1}{2\pi}\int_0^{2\pi}\frac{-\sin\phi(-\cos\phi)' - (-\cos\phi)(-\sin\phi)'}{\sin^2\phi + \cos^2\phi}d\phi = -\frac{1}{\pi}\int_0^{2\pi}d\phi = -2.$$

With $\sigma \equiv 0$ we have conditions of the theorem 4.5 from [1, p. 198]. So from this theorem we have that the infinitesimal MG–deformation is trivial only, with $\sigma \equiv 0$, i. e. $\vec{y} = \overrightarrow{\mathrm{const}}$. Thus, we proved the part a) of our theorem.

With $\sigma \not\equiv 0$ we have conditions of the theorem 4.12 from [1, p. 202], from the theorem 4.12 we get the part b) of our theorem.

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